

# Rates of mixing for nonMarkov infinite measure semiflows

Henk Bruin\*      Ian Melbourne†      Dalia Terhesiu‡

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## Abstract

We develop an abstract framework for obtaining optimal rates of mixing for infinite measure semiflows. Previously, such results were restricted to the Markov setting. As an illustration of the method, we consider mixing rates for suspensions over nonMarkov intermittent maps with infinite measure.

## 1 Introduction

Decay of correlations is a delicate phenomenon for continuous time dynamical systems. Exponential decay of correlations has been established for certain classes of Anosov flows [12, 22, 35], and the techniques have been extended to various (non)uniformly hyperbolic flows [3, 4, 5, 6, 7, 11]. Nevertheless, the class of flows for which exponential decay has been established is very restricted.

The situation for superpolynomial decay of correlations (rapid mixing) is somewhat better. Rapid mixing for (nontrivial) basic sets for typical Axiom A flows was established in [13, 15], and was extended in [25] to nonuniformly hyperbolic flows given by a suspension over a Young tower with exponential tails [36].

For slowly mixing nonuniformly hyperbolic flows (suspensions over Young towers with polynomial tails [37]), the method of [13, 25] was used in [26] to establish polynomial decay of correlations.

Recently [30] developed operator renewal theory for continuous time dynamical systems, extending the discrete time theory of [18, 33]. The framework in [30] applies

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\*Faculty of Mathematics, University of Vienna, 1090 Vienna, AUSTRIA

†Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

‡Mathematics Department, University of Exeter, EX4 4QF, UK

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to slowly mixing nonuniformly hyperbolic semiflows that can be modelled as suspensions over full branch Gibbs-Markov maps (uniformly expanding Markov maps with an at most countable Markov partition satisfying bounded distortion). For this class of continuous time systems, [30] shows that the polynomial decay rates in [26] are sharp.

The paper of [30] also addresses mixing and rates of mixing for infinite measure nonuniformly expanding semiflows, extending the work of [20, 28] in the discrete time setting. Again, the results in [30] are restricted to the Markov setting.

In the current paper, we introduce a functional analytic framework that dispenses with the Gibbs-Markov structure in [30]. An illustrative example to which our theory applies is the following.

**Example 1.1** Let  $X = [0, 1]$  and consider the map  $f : X \rightarrow X$  given by

$$f(x) = x(1 + c_1 x^{\gamma_1}) \bmod 1 \quad \text{where } \gamma_1 \in (1, 2), c_1 \in (0, 1]. \quad (1.1)$$

This is an example of an AFN map [38], namely a nonuniformly expanding one-dimensional map with at most countably (in this case finitely) many branches with finite images and satisfying Adler's distortion condition  $\text{ess sup } |f''|/|f'|^2 < \infty$ . Up to scaling, there is a unique absolutely continuous invariant measure  $\mu_X$ . The measure  $\mu_X$  is infinite and the density has a singularity at the neutral fixed point 0.

Let  $\tau_0 : [0, 1] \rightarrow [2, \infty)$  be a continuous roof function and let  $f_t$  denote the suspension semiflow on  $X^{\tau_0}$  with invariant measure  $\mu_X^{\tau_0} = \mu_X \times \text{Lebesgue}$ . Note that there is now a neutral periodic solution of period  $\tau_0(0)$ .

If  $c_1 = 1$ , then  $f$  is Markov and the semiflow  $f_t$  is covered by the framework in [30]. For  $c_1 \in (0, 1)$ , we are in the nonMarkov setting and new methods are required. Roughly speaking, we show that for any  $\epsilon > 0$  and for sufficiently regular roof function  $\tau_0$  and sufficiently regular observables  $v$  and  $w$  supported away from the neutral periodic solution, there exist constants  $d_1 > 0$  and  $d_2, d_3, \dots \in \mathbb{R}$  (typically nonzero), such that

$$\rho_{v,w}(t) = \sum_j d_j t^{-j(1-\beta)} \int v \int w + O(t^{-(\frac{1}{2}-\epsilon)}).$$

Here,  $\beta = 1/\gamma_1 \in (\frac{1}{2}, 1)$ , and the sum is over those  $j \geq 1$  with  $j(1-\beta) \leq \frac{1}{2} - \epsilon$ .

There are three ingredients (one new ingredient, as well as two recent ones) that make the generalisation to nonMarkov semiflows possible:

- (a) A certain family of renewal operators  $\hat{R}(s)$  arising in [30] cannot be usefully viewed as a Laplace transform. Consequently, numerous calculations in [30] were restricted to the Gibbs-Markov setting. The main new idea used to deal with the nonMarkov setting is to obtain a factorisation  $\hat{R}(s) = g(s)\hat{M}(s)$  where  $\hat{M}$  is a genuine Laplace transform and  $g$  is a simple scalar function (Lemma 4.3 below). This enables various functional analytic techniques used in the discrete

time setting to be easily employed for continuous time. We refer to Section 4 for details.

- (b) In Section 4, we also incorporate ideas from [24] (based on [21]) for dealing with perturbation theory of transfer operators, thereby significantly relaxing the functional-analytic hypotheses.
- (c) In Section 5, we incorporate the idea of using a second (reinduced) suspension semiflow model for the study of high Fourier modes. This method was introduced in [31] for the study of toral extensions of nonMarkov slowly mixing dynamical systems (with finite and infinite measure). As in [31], reinducing facilitates the use of Dolgopyat-type arguments.

The focus in this paper is on rates of mixing for infinite measure semiflows. As pointed out to us by Dima Dolgopyat, Péter Nándori and Doma Szász, mixing itself does not require Dolgopyat-type arguments, so the reinducing idea is not required for mixing without rates. However, ingredients (a) and (b) remain useful for studying mixing of nonMarkov infinite measure semiflows. This is the subject of work in progress. Also, we anticipate that ingredients (a) and (c) will form the basis of future work on correlation decay for nonMarkov finite measure semiflows, extending [30].

The remainder of the paper is organised as follows. In Section 2 we introduce the abstract functional analytic setting, and state our main result, Theorem 2.4. An outline of the proof of Theorem 2.4 is presented in Section 3. Sections 4 and 5 contain proofs of the main lemmas (Lemma 3.2 and 3.3) dealing with small and large Fourier modes respectively. Section 6 shows how to enlarge the class of observables  $v$  and  $w$  under stronger assumptions on the underlying dynamics. Finally, Section 7 shows how Example 1.1 fits into the abstract framework.

**Notation** We use “big O” and  $\ll$  notation interchangeably, writing  $a_n = O(b_n)$  or  $a_n \ll b_n$  if there is a constant  $C > 0$  such that  $a_n \leq Cb_n$  for all  $n \geq 1$ .

## 2 Abstract set-up

Let  $(Y, d_Y)$  be a bounded metric space with Borel probability measure  $\mu$  and let  $F : Y \rightarrow Y$  be an ergodic and mixing measure-preserving transformation. Let  $\tau : Y \rightarrow \mathbb{R}$  be a nonintegrable roof function bounded away from zero. For convenience, we suppose that  $\text{ess inf } \tau > 2$ . Throughout we assume that

$$\mu(y \in Y : \tau(y) > t) = \ell(t)t^{-\beta} \text{ where } \beta \in (\tfrac{1}{2}, 1] \text{ and } \ell \text{ is a measurable slowly varying function (so } \lim_{t \rightarrow \infty} \ell(\lambda t)/\ell(t) = 1 \text{ for all } \lambda > 0).$$

Define the suspension  $Y^\tau = \{(y, u) \in Y \times \mathbb{R} : 0 \leq u \leq \tau(y)\} / \sim$  where  $(y, \tau(y)) \sim (Fy, 0)$ . The suspension semiflow  $F_t : Y^\tau \rightarrow Y^\tau$  is given by  $F_t(y, u) = (y, u + t)$ ,

computed modulo identifications. The measure  $\mu^\tau = \mu \times \text{Lebesgue}$  is ergodic and  $F_t$ -invariant.

Next, let  $Z \subset Y$  be a subset of positive measure (possibly  $Z = Y$ ). Let  $\sigma : Z \rightarrow \mathbb{Z}^+$  be an inducing time (not necessarily the first return time) such that  $F^{\sigma(z)}(z) \in Z$  for all  $z \in Z$ , yielding the induced map  $G = F^\sigma : Z \rightarrow Z$ . Throughout we assume that

$$\mu(z \in Z : \sigma(z) > n) = O(e^{-cn}) \text{ for some } c > 0.$$

Define the induced roof function  $\varphi = \tau_\sigma : Z \rightarrow \mathbb{R}^+$ . Then we can form the suspension semiflow  $G_t : Z^\varphi \rightarrow Z^\varphi$  where  $Z^\varphi = \{(z, u) \in Z \times \mathbb{R} : 0 \leq u \leq \varphi(z)\} / \sim$  where  $(z, \varphi(z)) \sim (Gz, 0)$ , and  $G_t : Z^\varphi \rightarrow Z^\varphi$  is given by  $G_t(z, u) = (z, u + t)$ , computed modulo identifications.

**Assumptions on  $F$  and  $\tau$**  Let  $\mathbb{H} = \{\text{Re } s > 0\}$  and  $\overline{\mathbb{H}} = \{\text{Re } s \geq 0\}$ . Let  $R : L^1(Y) \rightarrow L^1(Y)$  denote the transfer operator for  $F : Y \rightarrow Y$ , that is  $\int_Y Rv w d\mu = \int_Y v w \circ F d\mu$ . Define the twisted transfer operators  $\hat{R}(s) : L^1(Y) \rightarrow L^1(Y)$ ,  $s \in \overline{\mathbb{H}}$ ,

$$\hat{R}(s)v = R(e^{-s\tau}v).$$

We assume that there exists a Banach space  $\mathcal{B}(Y)$  containing constant functions, with norm  $\|\cdot\|_{\mathcal{B}(Y)}$ , and that there exists  $\delta > 0$ ,  $\alpha_0 \in (0, 1)$  and  $C > 0$  such that

- (H1) (i)  $\mathcal{B}(Y)$  is compactly embedded in  $L^1(Y)$ .  
(ii)  $\|\hat{R}(s)^n v\|_{\mathcal{B}(Y)} \leq C(|v|_{L^1(Y)} + \alpha_0^n \|v\|_{\mathcal{B}(Y)})$  for all  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ ,  $v \in \mathcal{B}(Y)$ ,  $n \geq 1$ .

**Remark 2.1** Part (i) and part (ii) for  $s = 0$  are satisfied for mixing Gibbs-Markov maps (with  $\mathcal{B}(Y)$  a Hölder space) and mixing AFU maps [38] (with  $\mathcal{B}(Y)$  consisting of bounded variation functions).

**Remark 2.2** Condition (H1)(ii) is equivalent to the existence of constants  $\alpha_1 \in (0, 1)$ ,  $n_0 \geq 1$  and  $\tilde{C} > 0$  (with  $\alpha_1 = \alpha_0^{n_0}$ ) such that  $\|\hat{R}(s)\|_{\mathcal{B}(Y)} \leq \tilde{C}$  and  $\|\hat{R}(s)^{n_0} v\|_{\mathcal{B}(Y)} \leq \tilde{C}|v|_{L^1(Y)} + \alpha_1 \|v\|_{\mathcal{B}(Y)}$  for all  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ ,  $v \in \mathcal{B}(Y)$ .

In particular, if the family  $s \mapsto \hat{R}(s)$  of operators on  $\mathcal{B}(Y)$  is continuous at  $s = 0$ , then (H1)(ii) holds for sufficiently small  $\delta$  if and only if it holds for  $s = 0$ .

**Assumptions on  $G$  and  $\sigma$**  Let  $d_Z$  be a metric on  $Z$ . We assume

- (H2) There is an at most countable measurable partition  $\alpha$  of  $Z$  with  $\mu(a) > 0$  for all  $a \in \alpha$  such that  $\sigma$  is constant on partition elements. Moreover, there are constants  $\lambda > 1$ ,  $\eta \in (0, 1]$ ,  $C > 0$ , such that for each  $a \in \alpha$ ,

- (i)  $G = F^\sigma$  restricts to a (measure-theoretic) bijection from  $a$  onto  $Z$ .  
(ii)  $d_Y(F^q y, F^q y') \leq C d_Z(Gy, Gy')$  for all  $y, y' \in a$ ,  $0 \leq q < \sigma(a)$ .

(iii)  $p = \frac{d\mu|_Z}{d\mu|_{Z \circ G}}$  satisfies  $p(z) \leq C\mu(a)$  and  $|p(z) - p(z')| \leq C\mu(a)d_Z(Gz, Gz')^\eta$ , for all  $z, z' \in a$ .

(H3) There exists  $C > 0$  such that  $|\tau_q(z) - \tau_q(z')| \leq C(\inf_a \varphi)d_Z(Gz, Gz')^\eta$  for all  $a \in \alpha$ ,  $z, z' \in a$ ,  $q \leq \sigma(a)$ .

(H4) There exist two periodic orbits for  $G_t$  with periods  $p_1, p_2$  such that  $p_1/p_2$  is Diophantine. We require that these periodic orbits intersect  $Z$  only in the interior of partition elements.

**Remark 2.3** If  $F$  is Gibbs-Markov, then we can take  $Z = Y$  and  $\sigma \equiv 1$ , so condition (H2) is redundant and (H3) reduces to the case  $q = 1$ . Moreover (H1) holds with  $\mathcal{B}(Y)$  a Hölder space; see, for instance, [30, Proposition 3.5]).

**Observables** Let  $\eta \in (0, 1]$ . Define  $\tilde{Y} = Y \times [0, 1]$ . Given  $v : \tilde{Y} \rightarrow \mathbb{R}$ , we define

$$\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}, \quad |v|_{C^\eta} = \sup_{y, y' \in Y, y \neq y'} \sup_{u \in [0, 1]} |v(y, u) - v(y', u)|/d_Y(y, y')^\eta.$$

Let  $C^\eta(\tilde{Y})$  be the space of observables  $v : \tilde{Y} \rightarrow \mathbb{R}$  for which  $\|v\|_{C^\eta(\tilde{Y})} < \infty$ .

Next, let  $\mathcal{B}(Y)$  be the Banach space in (H1). Define  $\|v\|_{\mathcal{B}(\tilde{Y})} = \sup_{u \in [0, 1]} \|v(\cdot, u)\|_{\mathcal{B}(Y)}$ . Then  $\mathcal{B}(\tilde{Y})$  is the space consisting of those  $v \in L^1(\tilde{Y})$  with  $\|v\|_{\mathcal{B}(\tilde{Y})} < \infty$ .

If  $v \in \mathcal{B}(\tilde{Y}) \cap C^\eta(\tilde{Y})$ , then we define  $\|v\| = \|v\|_{\mathcal{B}(\tilde{Y})} + \|v\|_{C^\eta(\tilde{Y})}$ .

For  $w : Y \times (0, 1) \rightarrow \mathbb{R}$ ,  $m \geq 0$ , set  $|w|_{\infty, m} = \max_{j=0, \dots, m} |\partial_u^j w|_\infty$ . We write  $w \in L^{\infty, m}(\tilde{Y})$  if  $\text{supp } w \subset Y \times (0, 1)$  and  $|w|_{\infty, m} < \infty$ .

Define

$$\rho_{v, w}(t) = \int_{Y^\varphi} v \circ F_t d\mu^\tau.$$

We can now state our main result.

**Theorem 2.4** *Suppose that (H1)–(H4) hold. Suppose further that there exists  $\beta \in (\frac{1}{2}, 1)$ ,  $q \in (1, 2\beta)$ ,  $c > 0$ , such that  $\mu(\tau > t) = ct^{-\beta} + O(t^{-q})$  and  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  for some  $p > \frac{\beta}{2\beta - q}$ . Define  $\kappa = \min\{\beta(1 - q^{-1}(2\beta - 1)), q - \beta\}$ .*

*Then there exist constants  $d_1 = c^{-1} \frac{1}{\pi} \sin \beta\pi$ ,  $d_2, d_3, \dots \in \mathbb{R}$ , and for any  $\epsilon > 0$ , there exists  $m \geq 1$ , such that*

$$\rho_{v, w}(t) = \sum_j d_j t^{-j(1-\beta)} \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau + O(\|v\| \|w\|_{\infty, m} t^{-(\kappa - \epsilon)}),$$

*for all  $v \in \mathcal{B}(\tilde{Y}) \cap C^\eta(\tilde{Y})$ ,  $w \in L^{\infty, m}(\tilde{Y})$ ,  $t > 0$ . Here, the sum is over those  $j \geq 1$  with  $j(1 - \beta) \leq \kappa - \epsilon$ .*

*In particular, if  $\mu(\varphi > t) = ct^{-\beta} + O(t^{-2\beta})$ , then the mixing rate is of the form  $O(\|v\| \|w\|_{\infty, m} t^{-(\frac{1}{2} - \epsilon)})$ . If in addition  $\beta > \frac{3}{4}$  and  $d_2 \neq 0$ , then we obtain second order asymptotics and the mixing rate is sharp.*

**Remark 2.5** Mixing rates and higher order asymptotics for the case  $\beta = 1$  can be also obtained in this framework, along the lines of [28, Section 9.2]. We omit these issues here.

For completeness, we mention also the following mixing result. For  $\beta < 1$ , define  $\tilde{\ell} = \ell$  and  $d_\beta = \frac{1}{\pi} \sin \beta\pi$ . For  $\beta = 1$ , define  $\tilde{\ell}(t) = \int_1^t \ell(s)s^{-1} ds$  and  $d_\beta = 1$ .

**Theorem 2.6** *Suppose that (H1)–(H4) hold. Suppose further that  $\tau$  is nonintegrable with  $\mu(\tau > t) = \ell(t)t^{-\beta}$  where  $\beta \in (\frac{1}{2}, 1]$  and  $\ell$  is slowly varying, and that  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  for some  $p > \frac{\beta}{2\beta-1}$ .*

*Then there exists  $m \geq 1$  and  $a : (0, \infty) \rightarrow (0, \infty)$  with  $\lim_{t \rightarrow \infty} a(t) = 0$ , such that*

$$|\tilde{\ell}(t)t^{1-\beta}\rho_{v,w}(t) - d_\beta \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau| \leq a(t) \|v\| \|w\|_{\infty, m},$$

*for all  $v \in \mathcal{B}(\tilde{Y}) \cap C^\eta(\tilde{Y})$ ,  $w \in L^{\infty, m}(\tilde{Y})$ ,  $t > 0$ .*

## Semiflows on ambient manifolds

In applications, we are often given a semiflow  $\psi_t : M \rightarrow M$  on a finite-dimensional manifold  $M$ , with codimension one cross-section  $X$  and first hit time  $h : X \rightarrow \mathbb{R}^+$  and Poincaré map  $\psi : X \rightarrow X$ . Here  $h(x) > 0$  is least such that  $\psi_{h(x)}(x) \in X$  and  $\psi(x) = \psi_{h(x)}(x)$ .

We are particularly interested in the situation where  $\psi : X \rightarrow X$  possesses a conservative ergodic absolutely continuous infinite Borel measure  $\mu_X$ . In this case, we fix a subset  $Y \subset X$  with  $\mu_X(Y) \in (0, \infty)$ . Define the first return time  $r : Y \rightarrow \mathbb{Z}^+$  and the first return map  $F = \psi^r : Y \rightarrow Y$  with ergodic invariant probability measure  $\mu = (\mu_X|_Y)/\mu_X(Y)$ . The induced roof function  $\tau(y) = \sum_0^{r(y)-1} h(f^j y)$  and suspension semiflow  $F_t : Y^\tau \rightarrow Y^\tau$  is as defined above.

Let  $\pi_M : Y^\tau \rightarrow M$  denote the semiconjugacy between  $F_t$  and  $\psi_t$  given by  $\pi_M(y, u) = \psi_u y$ . We assume that the suspension semiflow  $F_t : Y^\tau \rightarrow Y^\tau$  falls into the abstract setting above. Let  $\tilde{M} = \bigcup_{t \in [0, 1]} \psi_t(Y) = \pi_M(\tilde{Y})$ . Let  $v, w : \tilde{M} \rightarrow \mathbb{R}$  be observables such that  $w \in L^{\infty, m}(\tilde{M})$  and  $v \circ \pi_M \in \mathcal{B}(\tilde{Y}) \cap C^\eta(\tilde{Y})$  where  $\mathcal{B}(Y)$  is the Banach space in (H1). Then it is immediate that Theorems 2.4 and 2.6 apply to  $\int_M v w \circ \psi_t d\mu_X$ .

**Remark 2.7** Suppose that  $v \in C^\eta(\tilde{M})$  and that there is a constant  $C > 0$  such that  $|v(\psi_u y) - v(\psi_u y')| \leq C d_M(y, y')$  for all  $y, y' \in Y$ ,  $u \in [0, 1]$ . Then it is automatic that  $v \circ \pi_M \in C^\eta(\tilde{Y})$ . Of course, the Banach space  $\mathcal{B}(Y)$  needs to be chosen so that  $v \circ \pi_M \in \mathcal{B}(\tilde{Y})$  for reasonable observables  $v : \tilde{M} \rightarrow \mathbb{R}$ .

### 3 Strategy of proof

Recall that  $F_t : Y^\tau \rightarrow Y^\tau$  is a suspension semiflow over a mixing map  $F : Y \rightarrow Y$  with roof function  $\tau : Y \rightarrow \mathbb{R}^+$ , and that  $G_t : Z^\varphi \rightarrow Z^\varphi$  is a suspension semiflow over  $G = F^\sigma : Z \rightarrow Z$  with roof function  $\varphi = \tau_\sigma : Z \rightarrow \mathbb{R}^+$ . Recall also that  $\mu^\tau = \mu \times \text{Lebesgue}$  is an ergodic  $F_t$ -invariant measure on  $Y^\tau$ .

Assumption (H2) guarantees that there is a unique ergodic  $G$ -invariant probability measure  $\mu_Z$  on  $Z$  that is absolutely continuous with respect to  $\mu|_Z$ . We obtain an ergodic  $G_t$ -invariant measure  $\mu_Z^\varphi = (\mu_Z \times \text{Lebesgue}) / \int_Z \sigma d\mu_Z$  on  $Z^\varphi$ .

The projection  $\pi : Z^\varphi \rightarrow Y^\tau$  given by  $\pi(z, u) = F_u(z, 0)$  defines a semiconjugacy between the suspension semiflows  $G_t : Z^\varphi \rightarrow Z^\varphi$  and  $F_t : Y^\tau \rightarrow Y^\tau$ . The following result, proved in Section 5, shows that  $\pi$  is measure-preserving.

**Proposition 3.1**  $\pi_* \mu_Z^\varphi = \mu^\tau$ .

It follows that

$$\rho_{v,w}(t) = \int_{Y^\tau} v w \circ F_t d\mu^\tau = \int_{Z^\varphi} \hat{v} \hat{w} \circ G_t d\mu_Z^\varphi \quad \text{where } \hat{v} = v \circ \pi, \hat{w} = w \circ \pi.$$

Let  $\hat{\rho}_{v,w}(s) = \int_0^\infty e^{-st} \rho_{v,w}(t) dt$  denote the Laplace transform of  $\rho_{v,w}(t)$ . This is analytic on  $\mathbb{H}$ . We are particularly interested in the behaviour of  $\hat{\rho}_{v,w}(s)$  for  $s = ib$  purely imaginary. As indicated in the introduction (ingredient (c)) the strategy in this paper is to analyse  $\hat{\rho}_{v,w}(ib)$  using the two different expressions for  $\hat{\rho}(ib)$ . For  $b$  in a neighborhood of 0 ( $b$  “small”), we use the representation  $\rho_{v,w}(t) = \int_{Y^\tau} v w \circ F_t d\mu^\tau$ . For  $b$  outside a neighborhood of 0 ( $b$  “large”) we use the representation  $\rho_{v,w}(t) = \int_{Z^\varphi} \hat{v} \hat{w} \circ G_t d\mu_Z^\varphi$ .

The resulting estimates are stated in Lemmas 3.2 and 3.3 below, and are proved in Sections 4 and 5.

In the remainder of this section, we state the key estimates for small and large  $b$  (Lemmas 3.2 and 3.3) and use them to prove Theorems 2.4 and 2.6. Except in the proof of Lemma 3.6 below, we write  $\rho(t)$  and  $\hat{\rho}(s)$ , suppressing the dependence on  $v$  and  $w$ .

We assume throughout the hypotheses of Theorem 2.6, namely that (H1)–(H4) hold, that  $\mu(\tau > t) = \ell(t)t^{-\beta}$  where  $\beta \in (\frac{1}{2}, 1]$  and  $\ell$  is slowly varying, and that  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  where  $p > \frac{\beta}{2\beta-1}$ . The additional conditions in Theorem 2.4 are not assumed except when explicitly mentioned.

For  $\beta < 1$ , let  $c_\beta = i \int_0^\infty e^{-i\sigma} \sigma^{-\beta} d\sigma$ .

**Lemma 3.2** (a) For all  $a > 0$ ,  $v \in \mathcal{B}(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \ell(t) t^{1-\beta} \int_0^{a/t} e^{ibt} \hat{\rho}(ib) db &= c_\beta^{-1} \int_0^a e^{i\sigma} \sigma^{-\beta} d\sigma \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau, \quad \beta < 1, \\ \lim_{t \rightarrow \infty} \tilde{\ell}(t) \int_0^{a/t} \cos bt \operatorname{Re} \hat{\rho}(ib) db &= \frac{\pi}{2} \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau, \quad \beta = 1. \end{aligned}$$



(b) For  $\beta \leq 1$ , there exists  $C, \delta > 0$ , such that

$$|\hat{\rho}(s)| \leq C\tilde{\ell}(1/|s|)^{-1}|s|^{-\beta}\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})},$$

for all  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ ,  $v \in \mathcal{B}(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ . If  $\beta = 1$ , then in addition

$$|\operatorname{Re} \hat{\rho}(ib)| \leq C\ell(1/|b|)\tilde{\ell}(1/|b|)^{-2}|b|^{-1}\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})},$$

for all  $b \in \mathbb{R}$ ,  $0 < |b| < \delta$ .

(c) For  $\beta \leq 1$ , there exists  $C, \delta > 0, \gamma > 1 - \beta$ , such that

$$|\hat{\rho}(i(b+h)) - \hat{\rho}(ib)| \leq C\{\tilde{\ell}(1/b)^{-2}b^{-2\beta}\tilde{\ell}(1/h)h^\beta + b^{-\beta}h^\gamma\}\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})},$$

for all  $0 < h < b < \delta$ ,  $v \in \mathcal{B}(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ .

(d) Suppose in addition that  $\mu(\tau > t) = ct^{-\beta} + O(t^{-q})$  where  $\beta \in (\frac{1}{2}, 1)$ ,  $q \in (1, 2\beta)$ ,  $c > 0$ , and  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  for some  $p > \frac{\beta}{2\beta-q}$ . Then part (c) holds with  $\gamma > q - \beta$ . Also, there are constants  $c_j \in \mathbb{C}$  with  $c_0 = c^{-1}c_\beta$  such that for all  $a \in (0, \delta t)$ ,

$$\begin{aligned} \int_0^{a/t} e^{ibt} \hat{\rho}(ib) db &= \sum_j c_j \int_0^{a/t} b^{-((j+1)\beta-j)} e^{ibt} db \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau \\ &\quad + O((a/t)^{1-2\beta+q} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}), \end{aligned}$$

where the sum is over those  $j \geq 0$  with  $(j+1)\beta - j \geq 2\beta - q$ .

**Lemma 3.3** Let  $\delta, \epsilon > 0$ . There exists  $C, \omega > 0$  such that

$$|\hat{\rho}(i(b+h)) - \hat{\rho}(ib)| \leq Cb^\omega h^{\beta-\epsilon} \|v\|_{C^\eta(\tilde{Y})} |w|_{L^\infty(\tilde{Y})},$$

for all  $0 < h < \delta < b$ ,  $v \in C^\eta(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ .

Lemmas 3.2 and 3.3 are proved in Sections 4 and 5 respectively. We now have the necessary prerequisites for completing the proof of Theorem 2.6.

**Proposition 3.4 (cf. [30, Proposition 6.2])** The analytic function  $\hat{\rho}$  on  $\mathbb{H}$  extends to a continuous function on  $\overline{\mathbb{H}} \setminus \{0\}$ , and

$$\rho(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ibt} \hat{\rho}(ib) db = \frac{1}{\pi} \int_0^{\infty} \operatorname{Re}(e^{ibt} \hat{\rho}(ib)) db = \frac{2}{\pi} \int_0^{\infty} \cos bt \operatorname{Re} \hat{\rho}(ib) db.$$



**Proof** For  $\beta < 1$ , this is the same as the proof of [30, Proposition 6.2] with [30, Proposition 6.1] replaced by Lemma 3.2(b). The main thing is to show that  $\lim_{\delta \rightarrow 0} \int_{\Gamma_\delta} e^{st} \hat{\rho}(s) ds = 0$  and  $\lim_{\delta \rightarrow 0} \int_{-\delta}^{\delta} e^{ibt} \hat{\rho}(ib) db = 0$  for each fixed  $t$ , where  $\Gamma_\delta = \{s = \delta e^{i\psi} : |\psi| \leq \pi/2\}$ .

The corresponding estimates for  $\beta = 1$  are as follows. By Lemma 3.2(b),  $|\int_{\Gamma_\delta} \cos st \operatorname{Re} \hat{\rho}(s) ds| \ll \int_{-\pi/2}^{\pi/2} e^{\delta t} \tilde{\ell}(1/\delta)^{-1} d\psi \ll \tilde{\ell}(1/\delta)^{-1}$ , and  $|\int_{-\delta}^{\delta} \cos bt \operatorname{Re} \hat{\rho}(ib) db| \ll \int_0^{\delta} \ell(1/b) \tilde{\ell}(1/b)^{-2} b^{-1} db = \tilde{\ell}(1/\delta)^{-1}$ . The result follows since  $\lim_{x \rightarrow \infty} \tilde{\ell}(x) = \infty$ .  $\blacksquare$

**Lemma 3.5 (cf. [30, Proposition 6.4])** *Let  $\frac{1}{2} < \beta' < \beta \leq 1$ . There exists  $C$ ,  $\delta > 0$ ,  $\gamma > 1 - \beta$ , such that for all  $a \geq 1$ ,  $t > (a + \pi)/\delta$ ,  $v \in \mathcal{B}(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ ,*

$$\left| \int_{a/t}^{\delta} e^{ibt} \hat{\rho}(ib) db \right| \leq C \{ \tilde{\ell}(t)^{-1} t^{-(1-\beta)} a^{-(2\beta'-1)} + t^{-\gamma} \} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}.$$

*Under the extra conditions in Lemma 3.2(d), we can choose  $\gamma > q - \beta$ .*

**Proof** Throughout, we suppress the factor  $\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}$ . Write

$$I = \int_{a/t}^{\delta} e^{ibt} \hat{\rho}(ib) db = - \int_{(a+\pi)/t}^{\delta+\pi/t} e^{ibt} \hat{\rho}(i(b - \pi/t)) db.$$

Then  $2I = I_1 + I_2 + I_3$ , where

$$\begin{aligned} I_1 &= - \int_{\delta}^{\delta+\pi/t} e^{ibt} \hat{\rho}(i(b - \pi/t)) db, & I_2 &= \int_{a/t}^{(a+\pi)/t} e^{ibt} \hat{\rho}(ib) db, \\ I_3 &= \int_{(a+\pi)/t}^{\delta} e^{ibt} (\hat{\rho}(ib) - \hat{\rho}(i(b - \pi/t))) db. \end{aligned}$$

Clearly  $I_1 = O(t^{-1})$ . By Lemma 3.2(b) and Potter's bounds (see for instance [9]),

$$\begin{aligned} |I_2| &\ll \int_{a/t}^{(a+\pi)/t} \tilde{\ell}(1/b)^{-1} b^{-\beta} db = \tilde{\ell}(t)^{-1} t^{-(1-\beta)} \int_a^{a+\pi} [\tilde{\ell}(t)/\tilde{\ell}(t/\sigma)] \sigma^{-\beta} d\sigma \\ &\ll \tilde{\ell}(t)^{-1} t^{-(1-\beta)} \int_a^{a+\pi} \sigma^{-\beta'} d\sigma \ll \tilde{\ell}(t)^{-1} t^{-(1-\beta)} a^{-\beta'}. \end{aligned}$$

By Lemma 3.2(c,d) with  $h = \pi/t$ ,

$$|I_3| \ll \tilde{\ell}(t) t^{-\beta} \int_{a/t}^{\infty} \tilde{\ell}(1/b)^{-2} b^{-2\beta} db + t^{-\gamma} \int_0^{\delta} b^{-\beta} db = I_{3,1} + I_{3,2}.$$

By Potter's bounds,

$$\begin{aligned} I_{3,1} &= \tilde{\ell}(t)^{-1} t^{-(1-\beta)} \int_a^\infty [\tilde{\ell}(t)/\tilde{\ell}(t/\sigma)]^2 \sigma^{-2\beta} d\sigma \ll \tilde{\ell}(t)^{-1} t^{-(1-\beta)} \int_a^\infty \sigma^{-2\beta'} d\sigma \\ &\ll \tilde{\ell}(t)^{-1} t^{-(1-\beta)} a^{-(2\beta'-1)}. \end{aligned}$$

Also,  $I_{3,2} \ll t^{-\gamma}$ . This completes the proof.  $\blacksquare$

**Lemma 3.6 (cf. [30, Proposition 6.5])** *For any  $\delta > 0$ ,  $\epsilon \in (0, \beta)$ , there exists  $C > 0$ ,  $m \geq 1$ , such that for all  $t > 0$ ,  $v \in C^\eta(\tilde{Y})$ ,  $w \in L^{\infty, m}(\tilde{Y})$ ,*

$$|\int_\delta^\infty e^{ibt} \hat{\rho}(ib) db| \leq C t^{-(\beta-\epsilon)} \|v\|_{C^\eta(\tilde{Y})} |w|_{L^{\infty, m}(\tilde{Y})}.$$

**Proof** Let  $\omega$  be as in Lemma 3.3. Choose  $m$  such that  $m > \omega + 1$ . By [30, Proposition 3.7],  $\hat{\rho}_{v,w}(s) = \hat{p}_m(s) + \hat{r}_m(s)$ , where  $\hat{p}_m(s)$  is a linear combination of  $s^{-j}$ ,  $j = 1, \dots, m$ , and  $\hat{r}_m(s) = s^{-m} \hat{\rho}_{v, \partial_u^m w}(s)$ .

By the proof of [30, Proposition 6.5],  $|\int_\delta^\infty e^{ibt} \hat{p}_m(ib) db| \ll t^{-1} |v|_{L^\infty(\tilde{Y})} |w|_{L^{\infty, m}(\tilde{Y})}$ .

By Proposition 3.4,  $\hat{r}_m$  is well-defined and continuous on  $\mathbb{H} \setminus \{0\}$ . By Lemma 3.3 with  $h = \pi/t$ ,

$$|\hat{r}_m(ib) - \hat{r}_m(i(b - \pi/t))| \ll b^{-(m-\omega)} t^{-(\beta-\epsilon)} \|v\|_{C^\eta(\tilde{Y})} |\partial_u^m w|_{L^\infty(\tilde{Y})}.$$

Suppressing the term  $\|v\|_{C^\eta(\tilde{Y})} |\partial_u^m w|_{L^\infty(\tilde{Y})}$ ,

$$\begin{aligned} |2 \int_\delta^\infty e^{ibt} \hat{r}_m(ib) db| &\leq \int_\delta^\infty |\hat{r}_m(ib) - \hat{r}_m(i(b - \pi/t))| db + \int_\delta^{\delta+\pi/t} |\hat{r}_m(i(b - \pi/t))| db \\ &\ll t^{-(\beta-\epsilon)} \int_\delta^\infty b^{-(m-\omega)} db + O(t^{-1}) = O(t^{-(\beta-\epsilon)}), \end{aligned}$$

where in the last inequality we have used that  $m > \omega + 1$ .  $\blacksquare$

**Proof of Theorem 2.4** A calculation (see for example [28, Proposition 9.5]) shows that for every  $j \geq 0$ , there exists  $C_j \in \mathbb{C}$ , with  $C_0 = \int_0^\infty e^{i\sigma} \sigma^{-\beta} d\sigma$ , such that  $\int_0^{a/t} b^{-(j+1)\beta-j} e^{ibt} db = C_j t^{-(j+1)(1-\beta)} (1 + O(a^{-(j+1)\beta-j}))$ .

Note that  $\text{Re}(c_\beta^{-1} C_0) = \sin \beta \pi$ . Choosing  $a = t^{1-q^{-1}\beta-\epsilon'}$ , we obtain from Lemma 3.2(d) that there are constants  $d_1 = c^{-1} \pi^{-1} \sin \pi \beta$  and  $d_2, d_3, \dots \in \mathbb{R}$  such that

$$\frac{1}{\pi} \text{Re} \left( \int_0^{a/t} e^{ibt} \hat{\rho}_{v,w}(ib) db \right) = \sum_j d_j t^{-j(1-\beta)} \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau + O(t^{-\beta(1-q^{-1}(2\beta-1)-\epsilon)}).$$

Recall that  $\beta > \beta' > 1/2$ . By Lemmas 3.5 and 3.6, there exist  $\gamma > q - \beta$  such that  $|\int_{a/t}^\infty e^{ibt} \hat{\rho}(ib) db| \ll t^{-(1-\beta)} a^{-(2\beta'-1)} + t^{-\gamma}$ . Hence

$$\left| \int_{a/t}^\infty e^{ibt} \hat{\rho}(ib) db \right| \ll t^{-\beta(1-q^{-1}(2\beta-1)-\epsilon)} + t^{-(q-\beta)}.$$

The result now follows from Proposition 3.4. ■

**Proof of Theorem 2.6** Recall that  $\beta > \beta' > 1/2$  and  $\gamma > 1 - \beta$ . By Lemmas 3.2(a), 3.5 and 3.6, for  $\beta < 1$ ,

$$\lim_{t \rightarrow \infty} \tilde{\ell}(t) t^{1-\beta} \int_0^\infty e^{ibt} \hat{\rho}(ib) db = c_\beta^{-1} \int_0^a e^{i\sigma} \sigma^{-\beta} d\sigma \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau + O(a^{-(2\beta'-1)}).$$

Since  $a \geq 1$  is arbitrary,

$$\lim_{t \rightarrow \infty} \tilde{\ell}(t) t^{1-\beta} \int_0^\infty e^{ibt} \hat{\rho}(ib) db = c_\beta^{-1} \int_0^\infty e^{i\sigma} \sigma^{-\beta} d\sigma \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau.$$

Since  $\operatorname{Re}(c_\beta^{-1} \int_0^\infty e^{i\sigma} \sigma^{-\beta} d\sigma) = \sin \beta\pi$ , the result for  $\beta < 1$  follows from Proposition 3.4.

The proof for  $\beta = 1$  is identical, working with  $\int_0^\infty \cos bt \operatorname{Re} \hat{\rho}(ib) db$  instead of  $\int_0^\infty e^{ibt} \hat{\rho}(ib) db$ . ■

## 4 Estimates for small $b$ : proof of Lemma 3.2

Throughout this section, we suppose that hypothesis (H1) holds, that  $\mu(\tau > t) = \ell(t)t^{-\beta}$  where  $\beta \in (\frac{1}{2}, 1]$  and  $\ell$  is slowly varying, and that  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  where  $p > \frac{\beta}{2\beta-1}$ .

### 4.1 Representation of $\hat{\rho}_{v,w}(ib)$ for small $b$

**Transfer operators** Recall that  $R : L^1(Y) \rightarrow L^1(Y)$  denotes the transfer operator for  $F : Y \rightarrow Y$ . Also, for  $s \in \overline{\mathbb{H}}$ , we have the families of operators  $\hat{R}(s)v = R(e^{-s\tau}v)$  on  $L^1(Y)$ . Note that  $\hat{R}$  is analytic on  $\mathbb{H}$  and well-defined on  $\overline{\mathbb{H}}$ .

Given observables  $v, w : \tilde{Y} \rightarrow \mathbb{R}$ , we define  $v_s, w_s : Y \rightarrow \mathbb{C}$  for  $s \in \mathbb{C}$ , setting

$$v_s(y) = \int_0^1 e^{su} v(y, u) du, \quad w_s(y) = \int_0^1 e^{-su} w(y, u) du.$$

Also, define

$$\hat{J}(s) = - \int_Y \int_0^1 \int_0^u e^{su} v(y, u) e^{-st} w(y, t) dt du d\mu.$$

Define  $\hat{T}(s) = (I - \hat{R}(s))^{-1}$  for  $s \in \mathbb{H}$ . Following [32],

$$\hat{\rho}_{v,w}(s) = \hat{J}(s) + \int_Y \hat{T}(s) v_s w_s d\mu, \tag{4.1}$$

for all  $v \in L^1(Y)$ ,  $w \in L^\infty(Y)$ ,  $s \in \mathbb{H}$ . (See Appendix A for a proof.)

**Proposition 4.1**  $\|v_s\|_{\mathcal{B}(Y)} \leq e^{|\operatorname{Re} s|} \|v\|_{\mathcal{B}(\tilde{Y})}$ , for all  $s \in \mathbb{C}$ , and  $\|v_{i(b+h)} - v_{ib}\|_{\mathcal{B}(Y)} \leq h \|v\|_{\mathcal{B}(\tilde{Y})}$  for all  $b, h \geq 0$ .

The same result hold with  $\mathcal{B}$  changed to  $L^p$ ,  $1 \leq p \leq \infty$ , and/or  $v$  changed to  $w$ .

**Proof** We have  $\|v_s\|_{\mathcal{B}(Y)} \leq \int_0^1 e^{\operatorname{Re} s u} \|v(\cdot, u)\|_{\mathcal{B}(Y)} du \leq e^{|\operatorname{Re} s|} \|v\|_{\mathcal{B}(\tilde{Y})}$ , and  $\|v_{i(b+h)} - v_{ib}\|_{\mathcal{B}(Y)} \leq \int_0^1 |e^{i(b+h)u} - e^{ibu}| \|v(\cdot, u)\|_{\mathcal{B}(Y)} du \leq h \int_0^1 u \|v\|_{\mathcal{B}(\tilde{Y})} du \leq h \|v\|_{\mathcal{B}(\tilde{Y})}$ . ■

**Proposition 4.2** (a)  $|\hat{J}(s)| \leq e^{|\operatorname{Re} s|} \|v\|_{L^1(\tilde{Y})} \|w\|_{L^\infty(\tilde{Y})}$  for all  $s \in \mathbb{C}$ ,  $v \in L^1(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ , and

(b)  $|\hat{J}(i(b+h)) - \hat{J}(ib)| \leq h \|v\|_{L^1(\tilde{Y})} \|w\|_{L^\infty(\tilde{Y})}$  for all  $b, h > 0$ ,  $v \in L^1(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ .

**Proof**

$$\begin{aligned} |\hat{J}(s)| &\leq \int_Y \int_0^1 \int_0^1 e^{\operatorname{Re} s(u-t)} |v(y, u)| |w(y, t)| dt du d\mu \\ &\leq e^{|\operatorname{Re} s|} \|w\|_\infty \int_Y \int_0^1 |v(y, u)| du d\mu = e^{|\operatorname{Re} s|} \|v\|_1 \|w\|_\infty. \end{aligned}$$

Similarly,

$$|\hat{J}(i(b+h)) - \hat{J}(ib)| \leq h \int_Y \int_0^1 \int_0^1 |u - t| |v(y, u)| |w(y, t)| dt du d\mu \leq h \|v\|_1 \|w\|_\infty. \quad \blacksquare$$

## 4.2 Relating $\hat{R}$ with a Laplace transform

In this subsection, we show that  $\hat{R}$  can be related to a genuine Laplace transform.

**Lemma 4.3** Set  $M(t)v = R(1_{\{t < \tau < t+1\}}v)$  and define  $\hat{M}(s) = \int_0^\infty M(t)e^{-st} dt$ . Let  $g(s) = \frac{s}{e^s - 1}$ . Then for  $s \in \overline{\mathbb{H}}$  such that  $|e^s - 1| > 0$ , we have

$$\hat{R}(s) = g(s)\hat{M}(s), \quad \hat{R}(0) = \hat{M}(0) = R.$$

**Proof** Compute that for all  $v \in L^1(Y)$  and  $w \in L^\infty(Y)$ ,

$$\begin{aligned} \int_Y \hat{M}(s)v w d\mu &= \int_Y \int_0^\infty R(1_{\{t < \tau < t+1\}}v) w e^{-st} dt d\mu = \int_Y \int_{\tau-1}^\tau v w \circ F e^{-st} dt d\mu \\ &= \frac{e^s - 1}{s} \int_Y e^{-s\tau} v w \circ F d\mu = \frac{e^s - 1}{s} \int_Y \hat{R}(s)v w d\mu. \end{aligned}$$

Therefore,  $\hat{R}(s) = g(s)\hat{M}(s)$ . For the second equality, note that  $g(s) = 1 + O(s)$ , as  $s \rightarrow 0$ . ■

**Proposition 4.4** *Let  $q \in [1, 2\beta)$ . Let  $p > \beta/(2\beta - q)$ ,  $p^{-1} + r^{-1} = 1$ . There exists  $C > 0$ ,  $\gamma > q - \beta$ , such that  $\tau^\gamma \in L^r$  and  $\int_0^\infty t^\gamma |M(t)v|_{L^1(Y)} dt \leq C|\tau^\gamma|_{L^r(Y)}|v|_{L^p(Y)}$  for all  $v \in L^p(Y)$ .*

**Proof** Note that  $r < \beta/(q - \beta)$  and  $\tau^\gamma \in L^r(Y)$  for  $\gamma < \beta/r$ . Hence  $\tau^\gamma \in L^r(Y)$  for all  $\gamma \in (q - \beta, \beta/r)$ . Now

$$\begin{aligned} \int_0^\infty t^\gamma |M(t)v|_1 dt &\leq \int_0^\infty t^\gamma |1_{\{\tau < t < \tau+1\}}v|_1 dt = \int_Y |v| \left\{ \int_0^\infty t^\gamma 1_{\{\tau < t < \tau+1\}} dt \right\} d\mu \\ &= \int_Y |v| \left\{ \int_\tau^{\tau+1} t^\gamma dt \right\} d\mu \ll \int_Y |v| \tau^\gamma d\mu \leq |v|_p |\tau^\gamma|_r, \end{aligned}$$

as required. ■

### 4.3 Estimates for $\hat{T}(s)$

Viewing  $\hat{R}(s)$ ,  $s \in \overline{\mathbb{H}}$ , as a family of operators from  $\mathcal{B}(Y)$  to  $L^\infty(Y)$ , we first study its continuity properties using Lemma 4.3 and Proposition 4.4.

**Lemma 4.5** *There exists  $C > 0$ ,  $\gamma > 1 - \beta$ , such that for all  $s_1, s_2 \in \overline{\mathbb{H}} \cap B_1(0)$ ,*

$$\|\hat{R}(s_1) - \hat{R}(s_2)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \leq C|s_1 - s_2|^\gamma.$$

**Proof** Choose  $p > \beta/(2\beta - 1)$  such that  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  and  $\gamma \in (1 - \beta, 1]$  as in Proposition 4.4 (with  $q = 1$ ). Then  $|\hat{M}(s_1)v|_1 \leq \int_0^\infty |M(t)v|_1 dt \ll |v|_p \ll \|v\|_{\mathcal{B}(Y)}$  and

$$\begin{aligned} |(\hat{M}(s_1) - \hat{M}(s_2))v|_1 &= \left| \int_0^\infty M(t)v(e^{-s_1 t} - e^{-s_2 t}) dt \right|_1 \\ &\ll |s_1 - s_2|^\gamma \int_0^\infty t^\gamma |M(t)v|_1 dt \ll |s_1 - s_2|^\gamma |v|_p \ll |s_1 - s_2|^\gamma \|v\|_{\mathcal{B}(Y)}, \end{aligned}$$

for all  $s_1, s_2 \in \overline{\mathbb{H}}$ .

Now restrict to  $s \in \overline{\mathbb{H}}$  with  $|s| \leq 1$ . By Lemma 4.3,  $\hat{R}(s) = g(s)\hat{M}(s)$  where  $|g(s)| \ll 1$  and  $|g(s_1) - g(s_2)| \ll |s_1 - s_2|$ . The result follows. ■

**Lemma 4.6** *There exists  $\delta \in (0, 1)$  and a continuous family  $\lambda(s)$ ,  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ , of simple eigenvalues for  $\hat{R}(s) : \mathcal{B}(Y) \rightarrow \mathcal{B}(Y)$  with  $\lambda(0) = 1$ . In addition,  $P(s)$  is a bounded linear operator on  $\mathcal{B}(Y)$  for all  $s \in \overline{\mathbb{H}} \cap B_\delta$  and  $\sup_{s \in \overline{\mathbb{H}} \cap B_\delta(0)} \|P(s)\|_{\mathcal{B}(Y)} < \infty$ .*

*Moreover, there exists  $C > 0$ ,  $\gamma > 1 - \beta$ , such that the corresponding family of spectral projections  $P(s)$  satisfies*

$$\|P(s_1) - P(s_2)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \leq C|s_1 - s_2|^\gamma,$$

*for all  $s_1, s_2 \in \overline{\mathbb{H}} \cap B_\delta(0)$ ,*

**Proof** We verify the hypotheses (2)–(5) of [21, Corollary 1], thereby obtaining the required estimates for the family  $P(s)$  (with a slightly reduced value of  $\gamma$  than in Lemma 4.5). Simplicity of the family of eigenvalues  $\lambda(s)$  is a consequence of  $F$  being mixing.

Hypothesis (2) is immediate since  $\|\hat{R}(s)\|_{L^1(Y)} \leq 1$  for all  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$  and assumption (H1)(ii) corresponds to hypothesis (3). Hypothesis (4) follows from (H1)(i),(ii), and hypothesis (5) follows from Lemma 4.5.  $\blacksquare$

Let  $\zeta(s)$  denote the corresponding family of eigenfunctions normalized so that  $\int_Y \zeta(s) d\mu = 1$ . In particular,  $\zeta(0) \equiv 1$  and  $P(0)v = \int_Y v d\mu$  for all  $v \in \mathcal{B}(Y)$ . Also, define the complementary projections  $Q(s) = I - P(s)$ . It is immediate that  $\zeta(s)$  and  $Q(s)$  inherit the estimate obtained for  $P(s)$ .

Recall that  $c_\beta = i \int_0^\infty e^{-i\sigma} \sigma^{-\beta} d\sigma$  for  $\beta < 1$ . Let  $c_\beta = 1$  when  $\beta = 1$ .

**Lemma 4.7 (cf. [30, Lemma 5.5])** *Fix  $\delta \in (0, 1)$  as in Lemma 4.6. Then*

$$\hat{T}(s) = c_\beta^{-1} \tilde{\ell}(1/|s|)^{-1} s^{-\beta} (P(0) + E(s)) \quad \text{for all } s \in \overline{\mathbb{H}} \cap B_\delta(0),$$

where  $E(s)$  is a family of operators satisfying  $\lim_{s \rightarrow 0} \|E(s)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = 0$ .

**Proof** For similar arguments we refer to, for instance, [2, 24, 28, 27, 30].

Let  $\zeta_1(s) = \zeta(s) - \zeta(0)$ . By Lemma 4.6, there exists  $\gamma > 1 - \beta$  such that  $|\zeta_1(s)|_{L^1(Y)} \ll |s|^\gamma$  for all  $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ .

Recall that  $\hat{R}(s)v = R(e^{-s\tau}v)$ . Following [19] (a simplification of [2]),

$$\lambda(s) = \int_Y \lambda(s) \zeta(s) d\mu = \int_Y R(e^{-s\tau} \zeta(s)) d\mu = \int_Y e^{-s\tau} d\mu + \chi(s), \quad (4.2)$$

where using Lemma 4.3,

$$\begin{aligned} \chi(s) &= \int_Y (\hat{R}(s) - \hat{R}(0)) \zeta_1(s) d\mu = \int_Y (g(s) \hat{M}(s) - g(0) \hat{M}(0)) \zeta_1(s) d\mu \\ &= g(s) \int_Y (\hat{M}(s) - \hat{M}(0)) \zeta_1(s) d\mu + (g(s) - g(0)) \int_Y \hat{M}(0) \zeta_1(s) d\mu \\ &= g(s) \int_Y \int_0^\infty (e^{-st} - 1) 1_{\{t < \tau < t+1\}} \zeta_1(s) dt d\mu + (g(s) - g(0)) \int_Y \zeta_1(s) d\mu. \end{aligned}$$

By Proposition B.1(a),  $1 - \int_Y e^{-s\tau} d\mu \sim c_\beta \tilde{\ell}(1/|s|) s^\beta$  as  $s \rightarrow 0$ . Also,

$$\begin{aligned} |\chi(s)| &\leq |\zeta_1(s)|_{L^1(Y)} \left( \int_Y \int_0^\infty 1_{\{t < \tau < t+1\}} |e^{-st} - 1| dt d\mu + |g(s) - g(0)| \right) \\ &\ll |s|^\gamma \left( \int_0^\infty \mu(t < \tau < t+1) |e^{-st} - 1| dt + |s| \right). \end{aligned}$$

A standard calculation (see for instance [30, Proof of Lemma 5.8]) shows that  $\int_0^\infty \mu(t < \tau < t+1) |e^{-st} - 1| dt \ll \tilde{\ell}(1/|s|) |s|^\beta$ . Hence  $|\chi(s)| \ll \tilde{\ell}(1/|s|) |s|^{\gamma+\beta}$ , and it follows that  $1 - \lambda(s) \sim c_\beta \tilde{\ell}(1/|s|) s^\beta$  as  $s \rightarrow 0$ . Hence  $(1 - \lambda(s))^{-1} \sim c_\beta^{-1} \tilde{\ell}(1/|s|)^{-1} s^{-\beta}$  as  $s \rightarrow 0$ .

Next,

$$\hat{T}(s) = (1 - \lambda(s))^{-1}P(s) + (I - \hat{R}(s))^{-1}Q(s) = (1 - \lambda(s))^{-1}(P(0) + E(s)),$$

where

$$E(s) = P(s) - P(0) + (1 - \lambda(s))(I - \hat{R}(s))^{-1}Q(s).$$

By (H1),  $\|(I - \hat{R}(s))^{-1}Q(s)\|_{\mathcal{B}(Y)} = O(1)$ . By Lemma 4.6,  $\|P(s) - P(0)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \ll |s|^\gamma$ . Hence  $\|E(s)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = O(|s|^\gamma) + O(\tilde{\ell}(1/|s|)|s|^\beta)$ . ■

**Lemma 4.8** *Let  $\beta = 1$ . Fix  $\delta \in (0, 1)$  as in Lemma 4.6. Then*

$$\operatorname{Re} \hat{T}(ib) = \frac{\pi}{2} \ell(1/|b|) \tilde{\ell}(1/|b|)^{-2} |b|^{-1} (P(0) + E(b)) \quad \text{for all } b \in \mathbb{R}, 0 < |b| < \delta,$$

where  $E(b)$  is a family of operators satisfying  $\lim_{b \rightarrow 0} \|E(b)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = 0$ .

**Proof** First, we note that

$$1 - \lambda(ib) \sim \tilde{\ell}(1/|b|)b, \quad \operatorname{Re}(1 - \lambda(ib)) \sim \frac{\pi}{2} \ell(1/|b|)|b|,$$

as  $b \rightarrow 0$ . Indeed the first estimate is explicit in the proof of Lemma 4.7, and the same argument combined with Proposition B.1(b) yields the second estimate. Hence

$$\operatorname{Re}((1 - \lambda(ib))^{-1}) = \operatorname{Re}(1 - \lambda(ib))|1 - \lambda(ib)|^{-2} \sim \frac{\pi}{2} \ell(1/|b|) \tilde{\ell}(1/|b|)^{-2} |b|^{-1}.$$

Arguing as in the proof of Lemma 4.7,

$$\begin{aligned} & \| \operatorname{Re} \hat{T}(ib) - \operatorname{Re}((1 - \lambda(ib))^{-1})P(0) \|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \\ & \ll |1 - \lambda(ib)|^{-1} \|P(ib) - P(0)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} + \|(I - \hat{R}(ib))^{-1}Q(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \\ & \ll \tilde{\ell}(1/|b|)^{-1} |b|^{-(1-\gamma)} + 1, \end{aligned}$$

for some  $\gamma > 0$ , completing the proof. ■

**Lemma 4.9 (cf. [24, Proposition 3.7])** *Fix  $\delta \in (0, 1)$  as in Lemma 4.6. Then there exists  $C > 0$ ,  $\gamma > 1 - \beta$ , such that for all  $0 < h < b < \delta$ ,*

$$\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \leq C \{ \tilde{\ell}(1/b)^{-2} b^{-2\beta} \tilde{\ell}(1/h) h^\beta + b^{-\beta} h^\gamma \}.$$

**Proof** Using (4.2), we write

$$\lambda(i(b+h)) - \lambda(ib) = \int_Y (e^{-i(b+h)\tau} - e^{-ib\tau}) d\mu + \chi(i(b+h)) - \chi(ib),$$



where

$$\begin{aligned}
\chi(i(b+h)) - \chi(ib) &= (g(i(b+h)) - g(ib)) \int_Y \zeta_1(i(b+h)) d\mu \\
&\quad + (g(ib) - g(0)) \int_Y (\zeta_1(i(b+h)) - \zeta_1(ib)) d\mu \\
&\quad + (g(i(b+h)) - g(ib)) \int_Y \int_0^\infty (e^{-i(b+h)t} - 1) 1_{\{t < \tau < t+1\}} \zeta_1(i(b+h)) dt d\mu \\
&\quad + g(ib) \int_Y \int_0^\infty (e^{-i(b+h)t} - e^{-ibt}) 1_{\{t < \tau < t+1\}} \zeta_1(i(b+h)) dt d\mu \\
&\quad + g(ib) \int_Y \int_0^\infty (e^{-ibt} - 1) 1_{\{t < \tau < t+1\}} (\zeta_1(i(b+h)) - \zeta_1(ib)) dt d\mu.
\end{aligned}$$

Using the estimates from the proof of Lemma 4.7, it is easy to see that

$$|\chi(i(b+h)) - \chi(ib)| \ll \tilde{\ell}(1/h)h^\beta b^\gamma + \tilde{\ell}(1/b)b^\beta h^\gamma,$$

for some  $\gamma > 1 - \beta$ .

Also,  $|\int_Y (e^{-i(b+h)\tau} - e^{-ib\tau}) d\mu| \ll \tilde{\ell}(1/h)h^\beta$  by the argument used in the proof of [16, Lemma 3.3.2]. Hence

$$|\lambda(i(b+h)) - \lambda(ib)| \ll \tilde{\ell}(1/h)h^\beta + \tilde{\ell}(1/b)b^\beta h^\gamma. \quad (4.3)$$

Next recall as in Lemma 4.7 that

$$\hat{T}(ib) = (1 - \lambda(ib))^{-1} P(ib) + (I - \hat{R}(ib))^{-1} Q(ib) = J_1(b) + J_2(b).$$

Using (4.3) and the estimates from the proof of Lemma 4.7, it is easy to see that  $|J_1(b+h) - J_1(b)| \ll \ell(1/b)^{-2} b^{-2\beta} \tilde{\ell}(1/h)h^\beta + \ell(1/b)^{-1} b^{-\beta} h^\gamma$ .

Since  $0 < h < b < \delta$ , we have  $\ell(1/b)^{-1} < b^{-\epsilon} \leq h^{-\epsilon}$  for any  $\epsilon > 0$ , so shrinking  $\gamma$  slightly

$$|J_1(b+h) - J_1(b)| \ll \tilde{\ell}(1/b)^{-2} b^{-2\beta} \tilde{\ell}(1/h)h^\beta + b^{-\beta} h^\gamma.$$

A more complicated argument in [24, Proposition 3.8] shows that  $|J_2(b+h) - J_2(b)| \ll h^{\beta-\epsilon} \leq h^\gamma$  for some  $\gamma > 1 - \beta$ , completing the proof.  $\blacksquare$

We can now complete the proof of Lemma 3.2(a,b,c).

**Proof of Lemma 3.2(a,b,c)** (a) By Propositions 4.1 and 4.2, and Lemma 4.7,  $|\hat{J}(ib)| \ll |v|_{L^1(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}$  and

$$\begin{aligned}
|\hat{T}(ib)v_{ib} w_{ib} - \hat{T}_{ib}v_0 w_0|_{L^1(Y)} &\leq |\hat{T}(ib)(v_{ib} - v_0) w_{ib}|_{L^1(Y)} + |\hat{T}(ib)v_0 (w_{ib} - w_0)|_{L^1(Y)} \\
&\leq \|\hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \{\|v_{ib} - v_0\|_{\mathcal{B}(Y)} |w_{ib}|_{L^\infty(Y)} + \|v_0\|_{\mathcal{B}(Y)} |w_{ib} - w_0|_{L^\infty(Y)}\} \\
&\ll \tilde{\ell}(1/b)^{-1} b^{1-\beta} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}.
\end{aligned}$$

Substituting into (4.1), we obtain for all  $\beta \in (0, 1]$  that

$$\hat{\rho}(ib) = \int_Y \hat{T}(ib)v_0 w_0 d\mu + O(\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}). \quad (4.4)$$

By Lemma 4.7, for  $\beta \in (0, 1)$ ,

$$\int_Y \hat{T}(ib) v_0 w_0 d\mu = c_\beta^{-1} \ell(1/b)^{-1} b^{-\beta} \int_Y (P(0) + E(ib)) v_0 w_0 d\mu,$$

where  $\|E(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = o(1)$  as  $b \rightarrow 0$ . Now  $\int_Y P(0) v_0 w_0 d\mu = \int_{Y^\tau} v d\mu^\tau \int_{Y^\tau} w d\mu^\tau$  and  $\int_Y |E(ib) v_0 w_0| d\mu \leq \|E(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}$ . Hence

$$\hat{\rho}_{v,w}(ib) = c_\beta^{-1} \ell(1/b)^{-1} b^{-\beta} (A + h(b)),$$

where  $A = \int_{\tilde{Y}} v d\mu^\tau \int_{\tilde{Y}} w d\mu^\tau$  and  $\lim_{b \rightarrow 0} h(b) = 0$ . Part (a) with  $\beta < 1$  follows from this via the dominated convergence theorem as in [28, Lemma 5.2].

When  $\beta = 1$ , we use (4.4) and Lemma 4.8 to obtain

$$\operatorname{Re} \hat{\rho}(ib) = \int_Y \operatorname{Re} \hat{T}(ib) v_0 w_0 d\mu = \frac{\pi}{2} g(b) (A + h(b)),$$

where  $g(b) = \ell(1/b)^{-1} \tilde{\ell}(1/b)^{-2} b^{-1}$  with antiderivative  $\tilde{\ell}(1/b)^{-1}$ . Write  $\cos bt \operatorname{Re} \hat{\rho}_{v,w}(ib) = \frac{\pi}{2} (I_1(b) + I_2(b, t))$  where

$$I_1(b) = g(b) (A + h(b)), \quad I_2(b, t) = (\cos bt - 1) g(b) (A + h(b)).$$

Let  $H(t) = \sup_{b \in [0, a/t]} |h(b)| = o(1)$  as  $t \rightarrow \infty$ . Then  $\tilde{\ell}(t) \int_0^{a/t} I_1(b) db = \tilde{\ell}(t) \tilde{\ell}(t/a)^{-1} (A + H(t)) \rightarrow A$ . Next,

$$\tilde{\ell}(t) \int_0^{a/t} I_2(b, t) db = \int_0^a \frac{\cos \sigma - 1}{\sigma} \frac{\tilde{\ell}(t)}{\tilde{\ell}(t/\sigma)} \frac{\ell(t)}{\tilde{\ell}(t/\sigma)} d\sigma.$$

By Potter's bounds, the integrand is dominated by  $\sigma^{-\epsilon}$  for any  $\epsilon > 0$ . By Karamata  $\ell(x) = o(\tilde{\ell}(x))$  as  $x \rightarrow \infty$ . Hence the integrand converges to zero pointwise and  $\tilde{\ell}(t) \int_0^{a/t} I_2(b, t) db \rightarrow 0$  as  $t \rightarrow \infty$ . This completes the proof of (a) for  $\beta = 1$ .

(b) For  $\beta \in (0, 1]$ , by (4.1), Propositions 4.1 and 4.2, and Lemma 4.7,

$$\begin{aligned} |\hat{\rho}(s)| &\leq |\hat{J}(s)| + \|\hat{T}(s)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v_s\|_{\mathcal{B}(Y)} |w_s|_{L^\infty(Y)} \\ &\ll \tilde{\ell}(1/|s|)^{-1} |s|^{-\beta} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}. \end{aligned}$$

For  $\beta = 1$ , by (4.4), Propositions 4.1 and 4.2, and Lemma 4.8,

$$\begin{aligned} |\operatorname{Re} \hat{\rho}(ib)| &\leq +\|\operatorname{Re} \hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v_0\|_{\mathcal{B}(Y)} |w_0|_{L^\infty(Y)} + O(\|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}) \\ &\ll \ell(1/|b|) \tilde{\ell}(1/|b|)^{-2} |b|^{-1} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}, \end{aligned}$$

(c) By (4.1), Propositions 4.1 and 4.2, and Lemmas 4.7 and 4.9,

$$\begin{aligned} |\hat{\rho}(i(b+h)) - \hat{\rho}(ib)| &\leq |\hat{J}(i(b+h)) - \hat{J}(ib)| \\ &\quad + \|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v_{i(b+h)}\|_{\mathcal{B}(Y)} |w_{i(b+h)}|_{L^\infty(Y)} \\ &\quad + \|\hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v_{i(b+h)} - v_{ib}\|_{\mathcal{B}(Y)} |w_{i(b+h)}|_{L^\infty(Y)} \\ &\quad + \|\hat{T}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v_{ib}\|_{\mathcal{B}(Y)} |w_{i(b+h)} - w_{ib}|_{L^\infty(Y)} \\ &\ll \{h + \tilde{\ell}(1/b)^{-2} b^{-2\beta} \tilde{\ell}(1/h) h^\beta + b^{-\beta} h^\gamma + \tilde{\ell}(1/b)^{-1} b^{-\beta} h\} \|v\|_{\mathcal{B}(\tilde{Y})} |w|_{L^\infty(\tilde{Y})}, \end{aligned}$$

yielding the required estimate.  $\blacksquare$

It remains to consider part (d) of Lemma 3.2. For the remainder of this section, we assume the hypotheses of Theorem 2.4. Since  $\mathcal{B}(Y)$  is embedded in  $L^p(Y)$  for some  $p > \beta/(2\beta - q)$ , it follows from Proposition 4.4 that we can take  $\gamma > q - \beta$  in Lemmas 4.5, 4.6 and 4.9. Without loss,  $\gamma \in (q - \beta, \beta)$ .

**Lemma 4.10** *Assume (H0). Fix  $\delta \in (0, 1)$  as in Lemma 4.6. Then there are constants  $c_j \in \mathbb{C}$  with  $c_0 = c^{-1}c_\beta$  such that*

$$\hat{T}(ib) = \sum_j c_j b^{-(j+1)\beta-j} P(0) + E(ib) \quad \text{for all } b \in B_\delta(0),$$

where  $E(ib)$  is a family of operators satisfying  $\|E(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = O(b^{-(2\beta-q)})$ .

**Proof** We continue from the proof of Lemma 4.7 with  $s = ib$ . The main thing that changes under the stronger conditions on  $\mu(\tau > t)$  is an improved estimate of  $\int_Y e^{-ib\varphi} d\mu$ . By [28, Lemma 3.2], there are constants  $e_1, e_2 \in \mathbb{C}$  such that  $\int_Y e^{-ib\varphi} d\mu = 1 - e_1 b^\beta + e_1 e_2 b + O(b^q)$ .

From the proof of Lemma 4.7 (with  $s = ib$ ) and Lemma 4.6, we know that  $1 - \lambda(ib) = \int_Y e^{-ib\varphi} d\mu + O(b^{\gamma+\beta})$ , where  $\gamma > q - \beta$ . Hence,  $1 - \lambda(ib) = e_1 b^\beta (1 - e_1^{-1} e_2 b^{1-\beta} + O(b^{q-\beta}))$ . Thus,

$$(1 - \lambda(ib))^{-1} = \sum_j c_j b^{-(j+1)\beta-j} + O(b^{-(2\beta-q)}),$$

for constants  $c_0, c_1, \dots \in \mathbb{C}$  with  $c_0 = c^{-1}c_\beta$ .

By the proof of Lemma 4.7,

$$\hat{T}(ib) = (1 - \lambda(ib))^{-1} (P(0) + \tilde{E}(ib)),$$

where  $\|\tilde{E}(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = O(b^\gamma) + O(b^\beta) = O(b^{q-\beta})$ .  $\blacksquare$

**Proof of Lemma 3.2(d)** As before,  $\hat{\rho}(ib) = \int_Y \hat{T}(ib) v_0 w_0 d\mu + O(\|v\|_{\mathcal{B}(\tilde{Y})} \|w\|_{L^\infty(\tilde{Y})})$ .

Also, by Lemma 4.10,

$$\int_Y \hat{T}(ib) v_0 w_0 d\mu = \sum_j c_j b^{-(j+1)\beta-j} \int_Y P(0) v_0 w_0 d\mu + \int_Y E(ib) v_0 w_0 d\mu,$$

where  $\|E(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} = O(b^{-(2\beta-q)})$ . Now  $\int_Y P(0) v_0 w_0 d\mu = \int_{Y^\tau} v d\mu^\tau \int_{Y^\tau} w d\mu^\tau$  and  $\int_Y |E(ib) v_0 w_0| d\mu \leq \|E(ib)\|_{\mathcal{B}(Y) \rightarrow L^1(Y)} \|v\|_{\mathcal{B}(\tilde{Y})} \|w\|_{L^\infty(\tilde{Y})}$ . Hence

$$\hat{\rho}_{v,w}(ib) = \sum_j c_j b^{-(j+1)\beta-j} \int_{\tilde{Y}} v d\tilde{\mu} \int_{\tilde{Y}} w d\tilde{\mu} + O(b^{-(2\beta-q)} \|v\|_{\mathcal{B}(\tilde{Y})} \|w\|_{L^\infty(\tilde{Y})}),$$

and the result follows since  $2\beta - q \in (0, 1)$ .  $\blacksquare$

## 5 Estimates for large $b$ : proof of Lemma 3.3

Throughout this section, we suppose that assumptions (H2)–(H4) hold.

### 5.1 Lifting from $Y^\tau$ to $Z^\varphi$

Recall that  $F_t : Y^\tau \rightarrow Y^\tau$  is the suspension semiflow over  $F$  with roof function  $\tau$  and  $G_t : Z^\varphi \rightarrow Z^\varphi$  is the suspension semiflow over  $G = F^\sigma$  with roof function  $\varphi = \tau_\sigma$ . Here,  $G : Z \rightarrow Z$  is a Gibbs-Markov map with partition  $\alpha$ . Also  $\pi : Z^\varphi \rightarrow Y^\tau$  is the semiconjugacy  $\pi(z, u) = F_u(z, 0)$ .

We defined ergodic  $F_t$ - and  $G_t$ -invariant measures  $\mu^\tau = \mu \times \text{Lebesgue}$  on  $Y^\tau$  and  $\mu_Z^\varphi = (\mu_Z \times \text{Lebesgue}) / \int_Z \sigma d\mu_Z$  on  $Z^\varphi$ . As promised, we now verify that  $\pi_* \mu_Z^\varphi = \mu^\tau$ .

**Proof of Proposition 3.1** The measures  $\mu_Z$  and  $\mu$  are related by the formula

$$\int_Y h d\mu = \bar{\sigma}^{-1} \int_Z \sum_{q=0}^{\sigma(z)-1} h(F^q z) d\mu_Z(z) \quad \text{for all } h \in L^1(Y),$$

where  $\bar{\sigma} = \int_Z \sigma d\mu_Z$ . For  $g \in L^1(Y^\tau)$ ,

$$\begin{aligned} \bar{\sigma} \int_{Y^\tau} g d\mu^\tau &= \bar{\sigma} \int_Y \int_0^{\tau(y)} g(y, u) du d\mu = \int_Z \sum_{q=0}^{\sigma(z)-1} \int_0^{\tau(F^q z)} g(F^q z, u) du d\mu_Z(z) \\ &= \int_Z \sum_{q=0}^{\sigma(z)-1} \int_{\tau_q(z)}^{\tau_{q+1}(z)} g(F^q z, u - \tau_q(z)) du d\mu_Z(z) \\ &= \int_Z \sum_{q=0}^{\sigma(z)-1} \int_{\tau_q(z)}^{\tau_{q+1}(z)} g(F_u(z, 0)) du d\mu_Z(z) = \int_Z \int_0^{\varphi(z)} g(F_u(z, 0)) du d\mu_Z(z) \\ &= \int_Z \int_0^{\varphi(z)} g \circ \pi(z, u) du d\mu_Z(z) = \bar{\sigma} \int_{Z^\varphi} g \circ \pi d\mu_Z^\varphi = \bar{\sigma} \int_{Y^\varphi} g d\pi_* \mu_Z^\varphi. \end{aligned}$$

Hence  $\pi_* \mu_Z^\varphi = \mu^\tau$  as required. ■

Define

$$\widehat{Y} = \pi^{-1}(\widetilde{Y}) = \bigcup_{z \in Z} \bigcup_{\ell=0}^{\sigma(z)-1} \{z\} \times [\tau_\ell(z), \tau_\ell(z) + 1].$$

We note that

$$\int_0^{\varphi(z)} 1_{\widehat{Y}}(z, u) du = \sigma(a) \quad \text{for all } a \in \alpha, z \in a. \quad (5.1)$$

Observables  $v : Y^\tau \rightarrow \mathbb{R}$  supported in  $\widetilde{Y}$  lift to observables  $\hat{v} = v \circ \pi : Z^\varphi \rightarrow \mathbb{R}$  supported in  $\widehat{Y}$ . Define  $\hat{v}_b : Z \rightarrow \mathbb{C}$ ,  $b \in \mathbb{R}$ ,

$$\hat{v}_b(z) = \int_0^{\varphi(z)} e^{ibu} \hat{v}(z, u) du = \int_0^{\varphi(z)} e^{ibu} (1_{\widehat{Y}} \hat{v})(z, u) du.$$

**Proposition 5.1** *Let  $\epsilon > 0$ . There exists  $C > 0$  such that*

$$|\hat{v}_b(z)| \leq \sigma(a)|v|_\infty, \quad |\hat{v}_b(z) - \hat{v}_b(z')| \leq C(b^\epsilon + 1)\sigma(a)(\inf_a \varphi^\epsilon)\|v\|_{C^\eta} d_Z(Gz, Gz')^\epsilon,$$

for all  $v \in C^\eta(\tilde{Y})$ ,  $a \in \alpha$ ,  $z, z' \in \alpha$ ,  $b > 0$ .

**Proof** By (5.1),  $|\hat{v}_b(z)| \leq |v|_\infty \int_0^{\varphi(z)} 1_{\hat{Y}}(z, u) du = \sigma(a)|v|_\infty$ . Next,

$$\begin{aligned} \hat{v}_b(z) &= \sum_{q=0}^{\sigma(a)-1} \int_{\tau_q(z)}^{\tau_q(z)+1} e^{ibu} \hat{v}(z, u) du = \sum_{q=0}^{\sigma(a)-1} \int_0^1 e^{ibu} e^{ib\tau_q(z)} \hat{v}(z, u + \tau_q(z)) du \\ &= \sum_{q=0}^{\sigma(a)-1} \int_0^1 e^{ibu} e^{ib\tau_q(z)} v(F^q z, u) du. \end{aligned}$$

Also, by (H2)(ii) and (H3),

$$\begin{aligned} \int_0^1 |e^{ib\tau_q(z)} v(F^q z, u) - e^{ib\tau_q(z')} v(F^q z', u)| du \\ \leq 2b^\epsilon |\tau_q(z) - \tau_q(z')|^\epsilon |v|_\infty + |v|_\eta d_Y(F^q z, F^q z')^\eta \\ \ll (b^\epsilon + 1)(\inf_a \varphi^\epsilon)\|v\|_{C^\eta(\tilde{Y})} d_Z(Gz, Gz')^\epsilon, \end{aligned}$$

for  $0 \leq q \leq \sigma(a) - 1$ . The estimate for  $|\hat{v}_b(z) - \hat{v}_b(z')|$  follows. ■

For  $v : \tilde{Z} \rightarrow \mathbb{C}$ ,  $\eta \in (0, 1)$ , define

$$\|v\|_{C^\eta(\tilde{Z})} = |v|_\infty + |v|_\eta, \quad |v|_\eta = \sup_{z, z' \in Z, z \neq z'} \sup_{u \in [0, 1]} |v(z, u) - v(z', u)| / d_Z(z, z')^\eta.$$

Define  $C^\eta(\tilde{Z})$  to be the space of observables  $v : \tilde{Z} \rightarrow \mathbb{C}$  such that  $\|v\|_{C^\eta(\tilde{Z})} < \infty$ .

## 5.2 Representation of $\hat{\rho}_{v,w}(ib)$ for large $b$

In this subsection we obtain an expression for  $\hat{\rho}_{v,w}(ib)$  using transfer operators related to the induced suspension semiflow  $G_t : Z^\varphi \rightarrow Z^\varphi$  over  $G = F^\sigma$  with roof function  $\varphi = \tau_\sigma$ .

Let  $L_t^G : L^1(Z^\varphi) \rightarrow L^1(Z^\varphi)$  denote the family of transfer operators corresponding to the suspension semiflow  $G_t : Z^\varphi \rightarrow Z^\varphi$ . Define  $\hat{L}^G(s) = \int_0^\infty L_t^G e^{-st} dt$  and note that for  $v \in L^1(\tilde{Y})$  and  $w \in L^\infty(\tilde{Y})$ ,

$$\hat{\rho}_{v,w}(s) = \int_{Z^\varphi} 1_{\hat{Y}} \hat{L}^G(s) (1_{\hat{Y}} \hat{v}) \hat{w} d\mu_Z^\varphi \quad \text{where } \hat{v} = v \circ \pi, \hat{w} = w \circ \pi. \quad (5.2)$$

Let  $\tilde{G} : \tilde{Z} \rightarrow \tilde{Z}$ ,  $\tilde{\varphi} : \tilde{Z} \rightarrow \mathbb{R}^+$  where  $\tilde{Z} = Z \times [0, 1]$ ,  $\tilde{G}(z, u) = (Gz, u)$  and  $\tilde{\varphi}(z, u) = \varphi(z)$ . Define  $T_t^G = 1_{\tilde{Z}} L_t^G 1_{\tilde{Z}}$  and let  $\hat{T}^G(s)$  be the corresponding Laplace transform.

**Proposition 5.2** *Let  $\epsilon \in (0, 1)$ ,  $\delta > 0$ . Then there exists  $C > 0$ ,  $\omega > 0$  such that*

$$\|\hat{T}^G(ib)\|_{C^{\epsilon\eta}(\tilde{Z}) \rightarrow L^\infty(\tilde{Z})} \leq Cb^\omega, \quad \|\hat{T}^G(i(b+h)) - \hat{T}^G(ib)\|_{C^{\epsilon\eta}(\tilde{Z}) \rightarrow L^\infty(\tilde{Z})} \leq Cb^\omega h^{\beta-\epsilon},$$

for all  $0 < h < \delta < b$ .

**Proof** Since  $G$  is Gibbs-Markov, the result with  $L^\infty(\tilde{Z})$  replaced by  $L^1(\tilde{Z})$  is covered by [30]. An additional but standard calculation (for the term  $\hat{U}$  in [30]) specialised to the Gibbs-Markov setting yields the  $L^\infty$  estimates.  $\blacksquare$

In a continuous-time analogue of [17], we define the operators

$$A_t : L^1(\tilde{Z}) \rightarrow L^1(Z^\varphi), \quad B_t : L^1(Z^\varphi) \rightarrow L^1(\tilde{Z})$$

where

$$A_t = 1_{Z_t} L_t^G 1_{\tilde{Z}}, \quad B_t = 1_{\tilde{Z}} L_t^G 1_{\Delta_t}, \quad E_t = L_t^G((1 - \xi(t))v).$$

Here

$$Z_t = \{(z, u) \in Z^\varphi : t \leq u \leq t+1, u > 1\},$$

$$\Delta_t = \{(z, u) \in Z^\varphi : \varphi(z) - t \leq u \leq \varphi(z) - t + 1\},$$

correspond to certain rows and diagonals in the suspension  $Z^\varphi$  and

$$\xi(t) = \int_0^t \int_0^x 1_{\Delta_y} 1_{Z_{t-x}} \circ G_t dy dx \in [0, 1].$$

Given Banach spaces  $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$  and families of operators  $\alpha_t : \mathcal{B}_2 \rightarrow \mathcal{B}_3$ ,  $\beta_t : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ ,  $t \geq 0$ , we define the convolution  $(\alpha \star \beta)_t : \mathcal{B}_1 \rightarrow \mathcal{B}_3$ ,  $t \geq 0$ , by  $(\alpha \star \beta)_t = \int_0^t \alpha_{t-x} \beta_x dx$ .

**Proposition 5.3**  $L_t^G = (A \star T^G \star B)_t + E_t$ , for all  $t \geq 0$ .

**Proof** For all  $0 \leq y \leq x \leq t$ ,

$$\begin{aligned} \int_{Z^\varphi} A_{t-x} T_{x-y}^G B_y v w d\mu_Z^\varphi &= \int_{Z^\varphi} 1_{Z_{t-x}} L_{t-x}^G (1_{\tilde{Z}} L_{x-y}^G (1_{\tilde{Z}} L_y^G (1_{\Delta_y} v))) w d\mu_Z^\varphi \\ &= \int_{Z^\varphi} 1_{Z_{t-x}} L_{t-x}^G (L_{x-y}^G (L_y^G (1_{\Delta_y} v))) w d\mu_Z^\varphi = \int_{Z^\varphi} 1_{Z_{t-x}} L_t^G (1_{\Delta_y} v) w d\mu_Z^\varphi \\ &= \int_{Z^\varphi} 1_{\Delta_y} v 1_{Z_{t-x}} \circ G_t w \circ G_t d\mu_Z^\varphi. \end{aligned}$$

Hence

$$\begin{aligned} \int_{Z^\varphi} (A \star T^G \star B)_t v w d\mu_Z^\varphi &= \int_{Z^\varphi} \int_0^t \int_0^x A_{t-x} T_{x-y}^G B_y v w dy dx d\mu_Z^\varphi \\ &= \int_{Z^\varphi} \xi(t) v w \circ G_t d\mu_Z^\varphi = \int_{Z^\varphi} L_t^G (\xi(t)) v w d\mu_Z^\varphi = \int_{Z^\varphi} L_t^G v w d\mu_Z^\varphi - \int_{Z^\varphi} E_t v w d\mu_Z^\varphi \end{aligned}$$

as required. ■

Let  $\hat{A}, \hat{B}, \hat{E}$  be the Laplace transforms of  $A_t, B_t, E_t$ . By Proposition 5.3,

$$\hat{L}^G(s) = \hat{A}(s)\hat{T}^G(s)\hat{B}(s) + \hat{E}(s). \quad (5.3)$$

Let  $\hat{B}(ib)1_{\hat{Y}}\pi_* : L^1(\tilde{Y}) \rightarrow L^1(\tilde{Z})$  denote the operator  $(\hat{B}(ib)1_{\hat{Y}}\pi_*)(v) = \hat{B}(ib)(1_{\hat{Y}}\hat{v})$  where  $\hat{v} = \pi_*v = v \circ \pi : \hat{Y} \rightarrow \mathbb{R}$ . Similarly, define  $1_{\hat{Y}}\hat{E}(ib)1_{\hat{Y}}\pi_* : L^1(\tilde{Y}) \rightarrow L^1(\hat{Y})$ . The following result is proved in Subsections 5.3 and 5.4.

**Lemma 5.4** *Let  $\epsilon \in (0, 1)$ ,  $\delta > 0$ . Then there exists  $C > 0$  such that*

- (a)  $\int_{t_0}^{\infty} \|1_{\hat{Y}}A_t\|_{L^\infty(\tilde{Z}) \rightarrow L^1(\hat{Y})} dt \leq Ct_0^{-(\beta-\epsilon)},$
- (b)  $\int_{t_0}^{\infty} \|1_{\hat{Y}}E_t1_{\hat{Y}}\pi_*\|_{L^\infty(\tilde{Y}) \rightarrow L^1(\hat{Y})} dt \leq Ct_0^{-(\beta-\epsilon)},$
- (c)  $\|\hat{B}(ib)1_{\hat{Y}}\pi_*\|_{C^\eta(\tilde{Y}) \rightarrow C^{\epsilon\eta}(\tilde{Z})} \leq Cb^\epsilon, \|\hat{B}(i(b+h))1_{\hat{Y}}\pi_* - \hat{B}(ib)1_{\hat{Y}}\pi_*\|_{C^\eta(\tilde{Y}) \rightarrow C^{\epsilon\eta}(\tilde{Z})} \leq Cb^\epsilon h^{\beta-2\epsilon},$

for all  $t_0 > 0$ ,  $0 < h < \delta < b$ .

**Proof of Lemma 3.3** It follows from Lemma 5.4 that in the appropriate norms,

$$\begin{aligned} \|1_{\hat{Y}}\hat{A}(ib)\| &\leq C, & \|1_{\hat{Y}}\hat{A}(i(b+h)) - 1_{\hat{Y}}\hat{A}(ib)\| &\leq Ch^{\beta-\epsilon}, \\ \|1_{\hat{Y}}\hat{E}(ib)1_{\hat{Y}}\pi_*\| &\leq C, & \|1_{\hat{Y}}\hat{E}(i(b+h))1_{\hat{Y}}\pi_* - 1_{\hat{Y}}\hat{E}(ib)1_{\hat{Y}}\pi_*\| &\leq Ch^{\beta-\epsilon}, \\ \|\hat{B}(ib)1_{\hat{Y}}\pi_*\| &\leq Cb^\epsilon, & \|\hat{B}(i(b+h))1_{\hat{Y}}\pi_* - \hat{B}(ib)1_{\hat{Y}}\pi_*\| &\leq Cb^\epsilon h^{\beta-2\epsilon}. \end{aligned}$$

Combining these with the estimates for  $\hat{T}^G$  in Proposition 5.2 and substituting into equation (5.3), there exist (new) constants  $C, \omega > 0$  such that

$$\|1_{\hat{Y}}\hat{L}^G(i(b+h))1_{\hat{Y}}\pi_* - 1_{\hat{Y}}\hat{L}^G(ib)1_{\hat{Y}}\pi_*\|_{C^\eta(\tilde{Y}) \rightarrow L^1(\hat{Y})} \leq Cb^\omega h^{\beta-2\epsilon}.$$

By (5.2),

$$\begin{aligned} |\hat{\rho}_{v,w}(i(b+h)) - \hat{\rho}_{v,w}(ib)| &\leq |\hat{w}|_\infty \|1_{\hat{Y}}\hat{L}^G(i(b+h))1_{\hat{Y}}\pi_*v - 1_{\hat{Y}}\hat{L}^G(ib)1_{\hat{Y}}\pi_*v\|_{L^1(\hat{Y})} \\ &\leq Cb^\omega h^{\beta-2\epsilon} \|v\|_{C^\eta(\tilde{Y})} |w|_\infty, \end{aligned}$$

for all  $v \in C^\eta(\tilde{Y})$ ,  $w \in L^\infty(\tilde{Y})$ . ■



### 5.3 Estimates for $1_{\widehat{Y}}A_t$ , $1_{\widehat{Y}}E_t1_{\widehat{Y}}\pi_*$ : proof of Lemma 5.4(a,b)

**Proof of Lemma 5.4(a)** Let  $t \geq 0$ . Then

$$|1_{\widehat{Y}}A_tv|_1 \leq \int_{Z^\varphi} 1_{\widehat{Y}}1_{Z_t}L_t^G(1_{\widetilde{Z}}|v|) d\mu_Z^\varphi \leq |v|_\infty \int_{\widetilde{Z}} 1_{\widehat{Y}} \circ G_t 1_{Z_t} \circ G_t d\mu_Z^\varphi,$$

Let  $(z, u) \in \widetilde{Z}$ . If  $G_t(z, u) \in Z_t$ , then  $u+t < \varphi(z)$  and  $G_t(z, u) = (z, u+t)$ . Hence

$$\|1_{\widehat{Y}}A_t\| \leq \int_{\widetilde{Z}} 1_{\widehat{Y}}(z, u+t) 1_{\{u+t \in [0, \varphi(z)]\}} d\mu_Z^\varphi.$$

Now,

$$\int_{t_0}^\infty 1_{\widehat{Y}}(z, u+t) 1_{\{u+t \in [0, \varphi(z)]\}} dt \leq 1_{\{\varphi(z) > t_0\}} \int_0^{\varphi(z)} 1_{\widehat{Y}}(z, t) dt \leq 1_{\{\varphi(z) > t_0\}} \sigma(z),$$

by (5.1), and so  $\int_{t_0}^\infty \|1_{\widehat{Y}}A_t\| dt \leq \int_Z 1_{\{\varphi > t_0\}} \sigma d\mu_Z$ .

Choose  $\epsilon' > 0$  and  $p, q > 1$  with  $p^{-1} + q^{-1} = 1$  such that  $(\beta - \epsilon')/p = \beta - \epsilon$ . Since  $\mu$  is  $G$ -invariant, by Hölder's inequality,

$$\int_{t_0}^\infty \|1_{\widehat{Y}}A_t\| dt \leq \mu(\varphi > t_0)^{1/p} |\sigma|_q \ll t_0^{-(\beta-\epsilon)},$$

completing the proof. ■

**Proof of Lemma 5.4(b)** It follows from the definitions that

$$|1_{\widehat{Y}}E_t(1_{\widehat{Y}}\pi_*v)|_1 \leq \int_{Z^\varphi} 1_{\widehat{Y}}L_t^G(|(1 - \xi(t))1_{\widehat{Y}}\hat{v}|) d\mu_Z^\varphi \leq |v|_\infty \int_{Z^\varphi} 1_{\widehat{Y}} \circ G_t |1 - \xi(t)| 1_{\widehat{Y}} d\mu_Z^\varphi.$$

It is convenient to break this expression into terms with the same lap number. Define  $\varphi_k = \sum_{j=0}^{k-1} \varphi \circ G^j$ . Given  $(z, u) \in Z^\varphi$  and  $t > 0$ , we define the lap number  $N_t = N_t(z, u)$  to be the integer  $N_t = k$  such that

$$u+t \in [\varphi_k(z), \varphi_{k+1}(z)).$$

Define  $E_{t,k}v = E_t(1_{\{N_t=k\}}v)$ . Then  $\|1_{\widehat{Y}}E_t1_{\widehat{Y}}\pi_*\| = \sum_{k=0}^\infty \|1_{\widehat{Y}}E_{t,k}1_{\widehat{Y}}\pi_*\|$  where

$$\|1_{\widehat{Y}}E_{t,k}1_{\widehat{Y}}\pi_*\| \leq \int_{Z^\varphi} 1_{\{N_t=k\}} 1_{\widehat{Y}} \circ G_t (1 - \xi(t)) 1_{\widehat{Y}} d\mu_Z^\varphi.$$

The desired result is immediate from the following two claims.

- (i) For each  $k \geq 0$ , there is a constant  $C_k > 0$  such that  $\int_{t_0}^\infty \|1_{\widehat{Y}}E_{t,k}1_{\widehat{Y}}\pi_*\| dt \leq C_k t_0^{-(\beta-\epsilon)}$ .

(ii)  $E_{t,k} = 0$  for all  $k \geq 2$ .

First, we prove claim (i). When  $N_t = k$ , we have

$$G_t(z, u) = (G^k z, u + t - \varphi_k(z)) \quad \text{and} \quad u + t - \varphi_k(z) \in [0, \varphi(G^k z)].$$

Hence,

$$\begin{aligned} \int_{t_0}^{\infty} \|1_{\widehat{Y}} E_{t,k} 1_{\widehat{Y}} \pi_*\| dt &\leq \int_{Z^\varphi} \int_{t_0}^{\infty} 1_{\{u+t \in [\varphi_k(z), \varphi_{k+1}(z)]\}} 1_{\widehat{Y}}(G^k z, u + t - \varphi_k(z)) dt 1_{\widehat{Y}} d\mu_Z^\varphi \\ &\leq \int_{Z^\varphi} 1_{\{\varphi_{k+1} > t_0\}} 1_{\widehat{Y}} \int_0^{\varphi(G^k z)} 1_{\widehat{Y}}(G^k z, t) dt d\mu_Z^\varphi \\ &= \int_{Z^\varphi} 1_{\{\varphi_{k+1} > t_0\}} 1_{\widehat{Y}} \sigma(G^k z) d\mu_Z^\varphi = \int_Z 1_{\{\varphi_{k+1} > t_0\}} \sigma \circ G^k \int_0^{\varphi(z)} 1_{\widehat{Y}}(z, u) du d\mu_Z \\ &= \int_Z 1_{\{\varphi_{k+1} > t_0\}} \sigma \circ G^k d\mu_Z, \end{aligned}$$

where we have used (5.1) twice. Using Hölder's inequality as in the proof of Lemma 5.4(a) completes the proof of claim (i).

To prove claim (ii), we show that  $\xi(t) \equiv 1$  when  $k \geq 2$ . The constraints  $(z, u) \in \Delta_y$ ,  $G_t(z, u) \in Z_{t-x}$  can be restated as

$$\varphi(z) - u < y < \varphi(z) - u + 1, \quad \varphi_k(z) - u < x < \varphi_k(z) - u + 1 < t.$$

These conditions alone imply that  $x, y > 0$ ,  $x < t$ . Since  $k \geq 2$ , we have also that  $y < \varphi(z) - u + 1 \leq \varphi_k(z) - u < x$ . Hence

$$\xi(t) = \int_0^t \int_0^x 1_{\Delta_y} 1_{Z_{t-x}} \circ G_t dy dx = \int_{\varphi_k(z)-u}^{\varphi_k(z)-u+1} \int_{\varphi(z)-u}^{\varphi(z)-u+1} 1 dy dx = 1,$$

as required. ■

## 5.4 Estimates for $\hat{B}(s)1_{\widehat{Y}}\pi_*$ : proof of Lemma 5.4(c)

For  $s \in \mathbb{C}$ , define the operator  $\hat{D}(s) : L^1(Y^\tau) \rightarrow L^1(\widetilde{Z})$ ,

$$(\hat{D}(s)v)(z, u) = e^{-s(\varphi(z)+u)} \int_0^{\varphi(z)} e^{st} \hat{v}(z, t) dt, \quad (z, u) \in \widetilde{Z}.$$

**Proposition 5.5**  $\hat{B}(s) = \tilde{R} \hat{D}(s)$ .

**Proof** Let  $v : Z^\varphi \rightarrow \mathbb{R}$ ,  $w : \tilde{Z} \rightarrow \mathbb{R}$ . We can regard  $w$  as a function on  $Z^\varphi$  supported on  $\tilde{Z}$ . Since  $G_t(z, u) = (Gz, u + t - \varphi(z))$  for  $(z, u) \in \Delta_t$ ,

$$\begin{aligned} \int_{\tilde{Z}} B_t v w d\mu_Z^\varphi &= \int_{Z^\varphi} L_t^G(1_{\Delta_t} v) w d\mu_Z^\varphi = \int_Z \int_0^{\varphi(z)} 1_{\{0 \leq u - \varphi(z) + t \leq 1\}} (v w \circ G_t)(z, u) du d\mu_Z \\ &= \int_Z \int_{t-\varphi(z)}^t 1_{\{0 \leq u \leq 1\}} v(z, u + \varphi(z) - t) w \circ \tilde{G}(z, u) du d\mu_Z \\ &= \int_Z \int_0^1 1_{\{t-\varphi(z) \leq u \leq t\}} v(z, u + \varphi(z) - t) w \circ \tilde{G}(z, u) du d\mu_Z. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\tilde{Z}} \hat{B}(s) v w d\mu_Z^\varphi &= \int_{\tilde{Z}} \int_u^{\varphi(z)+u} e^{-st} v(z, u + \varphi(z) - t) dt w \circ \tilde{G}(z, u) d\mu_Z^\varphi \\ &= \int_{\tilde{Z}} e^{-s(\varphi(z)+u)} \int_0^{\varphi(z)} e^{st} v(z, t) dt w \circ \tilde{G}(z, u) d\mu_Z^\varphi \\ &= \int_{\tilde{Z}} \hat{D}(s) v w \circ \tilde{G} d\mu_Z^\varphi = \int_{\tilde{Z}} \tilde{R} \hat{D}(s) v w d\mu_Z^\varphi, \end{aligned}$$

as required. ■

For  $b \in \mathbb{R}$ , define  $f(b) : L^1(\tilde{Y}) \rightarrow L^1(\tilde{Z})$ ,

$$f(b)v = \hat{D}(ib)(1_{\tilde{Y}} \pi_* v) = \hat{D}(ib)(1_{\tilde{Y}} \hat{v}), \quad \hat{v} = v \circ \pi.$$

**Proposition 5.6** *For any  $\epsilon \in (0, \beta]$ ,  $\delta > 0$ , there exists  $C > 0$  such that*

- (a)  $|(f(b)v)(z, u)| \leq \sigma(a)|v|_\infty$ ,
- (b)  $|(f(b)v)(z, u) - (f(b)v)(z', u)| \leq Cb^\epsilon \sigma(a)(\inf_a \varphi^\epsilon) \|v\|_{C^\eta(\tilde{Y})} d_Z(Gz, Gz')^{\epsilon\eta}$ ,
- (c)  $|\{f(b+h)v - f(b)v\}(z, u)| \leq Ch^{\beta-\epsilon} \sigma(a)(\inf_a \varphi^{\beta-\epsilon}) |v|_\infty$ ,
- (d)  $|\{f(b+h)v - f(b)v\}(z, u) - \{f(b+h)v - f(b)v\}(z', u)|$   
 $\leq Cb^\epsilon h^{\beta-2\epsilon} \sigma(a)(\inf_a \varphi^{\beta-\epsilon}) \|v\|_{C^\eta(\tilde{Y})} d_Z(Gz, Gz')^{\epsilon\eta}$ ,

for all  $a \in \alpha$ ,  $z, z' \in a$ ,  $u \in [0, 1]$ ,  $0 < h < \delta < b$ , and  $v \in C^\eta(\tilde{Y})$ .

**Proof** We establish the inequalities (a) and (b) together with

- (e)  $|(f'(b)v)(z, u)| \leq C\sigma(a)(\inf_a \varphi) |v|_\infty$ .
- (f)  $|(f'(b)v)(z, u) - (f'(b)v)(z', u)| \leq Cb^\epsilon \sigma(a)(\inf_a \varphi^{1+\epsilon}) \|v\|_{C^\eta(\tilde{Y})} d_Z(Gz, Gz')^{\epsilon\eta}$ .

By the mean value theorem, it follows from (e) that  $|\{f(b_1)v - f(b_2)v\}(z, u)| \ll \sigma(a)(\inf_a \varphi)|v|_\infty|b_1 - b_2|$ . Combining this with (a), we obtain

$$\begin{aligned} |\{f(b_1)v - f(b_2)v\}(z, u)| &\ll \sigma(a)|v|_\infty \min\{1, (\inf_a \varphi)|b_1 - b_2|\} \\ &\leq \sigma(a)|v|_\infty (\inf_a \varphi^{\beta-\epsilon})|b_1 - b_2|^{\beta-\epsilon}, \end{aligned}$$

proving (c). Similarly, using (b) and (f),

$$\begin{aligned} &|\{f(b_1)v - f(b_2)v\}(z, u) - \{f(b_1)v - f(b_2)v\}(z', u)| \\ &\ll (|b_1|^\epsilon + |b_2|^\epsilon)\sigma(a)(\inf_a \varphi^\epsilon)\|v\|_{C^\eta(\tilde{Y})}d_Z(Gz, Gz')^{\epsilon\eta} \min\{1, (\inf_a \varphi)|b_1 - b_2|\} \\ &\leq (|b_1|^\epsilon + |b_2|^\epsilon)\sigma(a)(\inf_a \varphi^{\beta-\epsilon})\|v\|_{C^\eta(\tilde{Y})}d_Z(Gz, Gz')^{\epsilon\eta}|b_1 - b_2|^{\beta-2\epsilon}, \end{aligned}$$

proving (d).

It remains to carry out the estimates (a), (b), (e), (f). In the notation of Proposition 5.1,

$$(f(b)v)(z, u) = e^{-ib(\varphi(z)+u)} \int_0^{\varphi(z)} e^{ibt}(1_{\hat{Y}}\hat{v})(z, t) dt = e^{-ib(\varphi(z)+u)}\hat{v}_b(z).$$

(a) By Proposition 5.1,  $|(f(b)v)(z, u)| = |\hat{v}_b(z)| \leq \sigma(a)|v|_\infty$ .

(e) Write

$$\begin{aligned} (f'(b)v)(z, u) &= -i(\varphi(z) + u)(f(b)v)(z, u) + i(h(b)v)(z, u), \\ (h(b)v)(z, u) &= e^{-ib(\varphi(z)+u)} \int_0^{\varphi(z)} te^{ibt}(1_{\hat{Y}}\hat{v})(z, t) dt. \end{aligned}$$

Since  $h$  differs from  $f$  only by a factor of  $t$  inside the integral, it is immediate that

$$|(h(b)v)(z, u)| \leq (\sup_a \varphi)|(f(b)v)(z, u)| \leq (\sup_a \varphi)\sigma(a)|v|_\infty \ll (\inf_a \varphi)\sigma(a)|v|_\infty.$$

Hence

$$|(f'(b)v)(z, u)| \leq ((\sup_a \varphi) + 1)|(f(b)v)(z, u)| + |(h(b)v)(z, u)| \ll \sigma(a)(\inf_a \varphi)|v|_\infty.$$

(b) Write  $(f(b)v)(z, u) - f(b)v(z', u) = I_1 + I_2$  where

$$I_1 = e^{-iu}(e^{-ib\varphi(z)} - e^{-ib\varphi(z')})\hat{v}_b(z), \quad I_2 = e^{-iu}e^{-ib\varphi(z')}(\hat{v}_b(z) - \hat{v}_b(z')).$$

By (H3) and Proposition 5.1,

$$\begin{aligned} |I_1| &\leq 2b^\epsilon|\varphi(z) - \varphi(z')|^\epsilon|v|_\infty\sigma(a) \ll b^\epsilon|v|_\infty\sigma(a)(\inf_a \varphi^\epsilon)d_Z(Gz, Gz')^{\epsilon\eta}, \\ |I_2| &= |\hat{v}_b(z) - \hat{v}_b(z')| \ll b^\epsilon\sigma(a)(\inf_a \varphi^\epsilon)\|v\|_{C^\eta(\tilde{Y})}d_Z(Gz, Gz')^{\epsilon\eta}. \end{aligned}$$

(f) Comparing the formulas for  $f$  and  $h$ , it is immediate that

$$\begin{aligned} |(h(b)v)(z, u) - (h(b)v)(z', u)| &\leq (\sup_a \varphi) |(f(b)v)(z, u) - (f(b)v)(z', u)| \\ &\ll (\inf_a \varphi) |(f(b)v)(z, u) - (f(b)v)(z', u)|. \end{aligned}$$

Hence

$$\begin{aligned} |(f'(b)v)(z, u) - f'(b)v)(z', u)| &\leq |\varphi(z) - \varphi(z')| |(f(b)v)(z, u)| \\ &\quad + |\varphi(z') + 1| |(f(b)v)(z, u) - (f(b)v)(z', u)| + |(h(b)v)(z, u) - (h(b)v)(z', u)| \\ &\ll b^\epsilon \sigma(a) (\inf_a \varphi^{1+\epsilon}) \|v\|_{C^\eta(\tilde{Y})} d_Z(Gz, Gz')^{\epsilon\eta} \end{aligned}$$

completing the proof. ■

**Proposition 5.7**  $\int_Z \sigma \varphi^{\beta-\epsilon} d\mu_Z < \infty$  for any  $\epsilon > 0$ .

**Proof** Let  $\delta > 0$ ,  $q \geq 1$ . Then

$$\mu_Z(\sigma \varphi^{\beta-\epsilon} > t) \leq \mu_Z(\sigma \varphi^{\beta-\epsilon} > t, \sigma \leq q \log t) + \mu_Z(\sigma > q \log t) = g(t) + O(t^{-cq}),$$

where

$$\begin{aligned} g(t) &= \mu_Z(\sigma \varphi^{\beta-\epsilon} > t, \sigma \leq q \log t) \leq \mu_Z(\varphi > ct^{(1-\delta)/(\beta-\epsilon)}, \sigma \leq q \log t) \\ &\leq \mu_Z(\tau_{[q \log t]} > ct^{(1-\delta)/(\beta-\epsilon)}) \leq \sum_{j=0}^{[q \log t]-1} \mu_Z(\tau \circ F^j > c't^{(1-2\delta)/(\beta-\epsilon)}) \\ &= [q \log t] \mu_Z(\tau > c't^{(1-2\delta)/(\beta-\epsilon)}) \ll t^{-(\beta-\delta)(1-2\delta)/(\beta-\epsilon)}. \end{aligned}$$

(Here,  $c, c' > 0$  are constants.) Choosing  $q$  sufficiently large and  $\delta$  sufficiently small, we obtain that  $\mu_Z(\varphi^{\beta-\epsilon} \sigma > t) = O(t^{-r})$  for some  $r > 1$  and the result follows. ■

**Proof of Lemma 5.4(c)** Using the standard formula for  $\tilde{R}$ , we obtain  $(\hat{B}(s)v)(z, u) = \sum_{a \in \alpha} p(z_a) (\hat{D}(s)v)(z_a, u)$ . In particular, for  $v \in C^\eta(\tilde{Y})$ ,  $\hat{v} = v \circ \pi$ ,

$$(\hat{B}(ib)1_{\tilde{Y}}\pi_*)v)(z, u) = (\hat{B})(ib)\hat{v})(z, u) = \sum_{a \in \alpha} p(z_a) (f(b)v)(z_a, u).$$

By (H2)(iii) and Propositions 5.6(a,b) and 5.7, for all  $z, z' \in \tilde{Z}$ ,  $u \in [0, 1]$ ,

$$|(\hat{B}(ib)\hat{v})(z, u)| \leq \sum_{a \in \alpha} p(z_a) |(f(b)v)(z_a, u)| \ll |v|_\infty \sum_{a \in \alpha} \mu_Z(a) \sigma(a) \ll |v|_\infty,$$

and

$$\begin{aligned} |(\hat{B}(ib)\hat{v})(z, u) - (\hat{B}(ib)\hat{v})(z', u)| &\leq \sum_{a \in \alpha} |p(z_a) - p(z'_a)| |(f(b)v)(z_a, u)| + \sum_{a \in \alpha} p(z'_a) |(f(b)v)(z_a, u) - (f(b)v)(z'_a, u)| \\ &\ll \sum_{a \in \alpha} \mu_Z(a) d_Z(Gz_a, Gz'_a)^\eta \sigma(a) |v|_\infty + \sum_{a \in \alpha} \mu_Z(a) b^\epsilon \sigma(a) (\inf_a \varphi^\epsilon) \|v\|_{C^\eta} d_Z(Gz_a, Gz'_a)^{\epsilon\eta} \\ &\ll b^\epsilon \|v\|_{C^\eta} d_Z(z, z')^{\epsilon\eta}. \end{aligned}$$

Hence  $\|\hat{B}(ib)\hat{v}\|_{C^{\epsilon\eta}(\tilde{Z})} \ll b^\epsilon \|v\|_{C^\eta(\tilde{Y})}$ .

Similarly, using (H2)(iii) and Propositions 5.6(e,f) and 5.7,

$$\begin{aligned} |\{\hat{B}(i(b+h))\hat{v} - \hat{B}(ib)\hat{v}\}(z, u)| &\leq \sum_{a \in \alpha} p(z_a) |\{f(b+h)v - f(b)v\}(z_a, u)| \\ &\ll \sum_{a \in \alpha} \mu_Z(a) h^{\beta-\epsilon} \sigma(a) (\inf_a \varphi^{\beta-\epsilon}) |v|_\infty \ll h^{\beta-\epsilon} |v|_\infty, \end{aligned}$$

and

$$\begin{aligned} &|\{\hat{B}(i(b+h))\hat{v} - \hat{B}(ib)\hat{v}\}(z, u) - \{\hat{B}(i(b+h))\hat{v} - \hat{B}(ib)\hat{v}\}(z', u)| \\ &\leq \sum_{a \in \alpha} |p(z_a) - p(z'_a)| |\{f(b+h)v - f(b)v\}(z, u)| \\ &\quad + \sum_{a \in \alpha} p(z'_a) |\{f(b+h)v - f(b)v\}(z, u) - \{f(b+h)v - f(b)v\}(z', u)| \\ &\ll \sum_{a \in \alpha} \mu_Z(a) d_Z(Gz_a, Gz'_a)^\eta h^{\beta-\epsilon} \sigma(a) (\inf_a \varphi^{\beta-\epsilon}) |v|_\infty \\ &\quad + \sum_{a \in \alpha} \mu_Z(a) b^\epsilon h^{\beta-2\epsilon} \sigma(a) \inf_a \varphi^{\beta-\epsilon} \|v\|_{C^\eta} d_Z(Gz_a, Gz'_a)^{\epsilon\eta} \\ &\ll b^\epsilon h^{\beta-2\epsilon} \|v\|_{C^\eta} d_Z(z, z')^{\epsilon\eta}. \end{aligned}$$

Hence  $\|\hat{B}(i(b+h))\hat{v} - \hat{B}(ib)\hat{v}\|_{C^{\epsilon\eta}(\tilde{Z})} \ll b^\epsilon h^{\beta-2\epsilon} \|v\|_{C^\eta(\tilde{Y})}$ . ■

## 6 Piecewise $C^{1+\eta}$ semiflows

In this section, we strengthen the conditions on the reinduced semiflow  $G_t : Z^\varphi \rightarrow Z^\varphi$  and weaken the assumptions on the observables  $v$  and  $w$ .

We continue to assume the tail conditions on  $\tau : Y \rightarrow \mathbb{R}^+$ ,  $\varphi : Z \rightarrow \mathbb{R}^+$  and  $\sigma : Z \rightarrow \mathbb{Z}^+$  as well as assumption (H1). As usual we require that  $\inf \varphi \geq 2$ .

Assumption (H2) is replaced by the following: Fix  $\eta \in (0, 1]$ . Let  $\{(c_m, d_m)\}$  be a countable partition mod 0 of  $Z = [0, 1]$  and suppose that  $G : Z \rightarrow Z$  is  $C^{1+\eta}$  on each subinterval  $(c_m, d_m)$  and extends to a homeomorphism from  $[c_m, d_m]$  onto  $Z$ . (In [4], the map is denoted by  $F$  and is  $C^{1+\alpha}$  but it is convenient here to use  $\eta$  instead of  $\alpha$ .) Suppose that  $\sigma$  is constant on partition elements of  $Z$  and that the roof function  $\varphi : Z \rightarrow \mathbb{R}^+$  is  $C^1$  on partition elements.

Let  $\mathcal{G}_n$  denote the set of inverse branches for  $G^n$  and write  $\mathcal{G} = \mathcal{G}_1$ . We assume that there are constants  $C > 0$ ,  $\rho_0 \in (0, 1)$  such that

- $|g'|_\infty \leq C\rho_0^n$  for all  $g \in \mathcal{G}_n$ ,
- $|\log |g'|_\eta| \leq C$  for all  $g \in \mathcal{G}$ ,

- $|(\varphi \circ g)'|_\infty \leq C$  for all  $g \in \mathcal{G}$ .

These correspond to conditions (i)–(iii) from [4]. In place of (H3), we require that

- $|\tau_q \circ g|_\eta \leq C|\varphi \circ g|_\infty$  for all  $q \leq \sigma \circ g$  and all  $g \in \mathcal{G}$ .

Finally, (H4) is replaced by the uniform nonintegrability condition

(UNI) For all  $n_0 \geq 1$ , there exists  $n \geq n_0$  and  $g_1, g_2 \in \mathcal{G}_n$  such that  $\psi = \varphi_n \circ g_1 - \varphi_n \circ g_2$  satisfies  $\inf |\psi'| > 0$ . (Here,  $\varphi_n = \sum_{j=0}^{n-1} \varphi \circ G^j$ .)

The main result in this section is:

**Theorem 6.1** *Under the above conditions, Theorems 2.4 and 2.6 hold with  $m = 2$ .*

**Remark 6.2** The set up in [4] extends results of [8] from  $C^2$  maps to  $C^{1+\alpha}$  maps. Higher-dimensional  $C^2$  maps were treated in [5]. Hence it is likely that Theorem 6.1 holds for semiflows over  $C^{1+\alpha}$  piecewise expanding maps in arbitrary dimensions.

The proof of Theorem 6.1 occupies the remainder of the section. First, we define the family of equivalent norms on  $C^\eta(Z)$ :

$$\|v\|_b = \max\{|v|_\infty, |v|_\eta/(1 + |b|^\eta), \quad b \in \mathbb{R}.$$

If  $L$  is an operator on  $C^\eta(Z)$ , we write  $\|L\|_b = \sup_{v \in C^\eta, \|v\|_b=1} \|Lv\|_b$ .

Let  $\hat{R}^G$  denote the transfer operator corresponding to  $G : Z \rightarrow Z$ , and define  $\hat{R}^G(s)v = R^G(e^{-s\varphi}v)$ .

There are various families of transfer operators  $P_s$ ,  $L_s$  and  $Q_s$  in [4]. We are only interested in the range  $s \in \overline{\mathbb{H}}$ . Moreover, we are particularly interested in  $s = ib$  imaginary. It is easily checked that  $L_{ib} = Q_{ib} = \hat{R}^G(ib)$ . In what follows, we use the notation  $\hat{R}^G(ib)$ . Also, the operator  $P_s$  in [4] satisfies  $P_s = f_0 \hat{R}^G(ib) f_0^{-1}$  where  $f_0, f_0^{-1} \in C^\eta$ .

The estimates in [4] are carried out for  $\operatorname{Re} s \geq -\epsilon$ ; it is easy to check that the estimates for  $s \in \overline{\mathbb{H}}$  hold even though condition (iv) from [4] is not assumed.

**Proposition 6.3** *There exists  $C > 0$  such that  $\|\hat{R}^G(ib)^n\|_b \leq C$  for all  $b \in \mathbb{R}$ ,  $n \geq 1$ .*

**Proof** See [4, Corollary 2.8]. ■

**Lemma 6.4** *There exist constants  $A, D > 0$  and  $\gamma \in (0, 1)$  such that*

$$\|\hat{R}^G(ib)^n\|_b \leq \gamma^n \text{ for all } n \geq A \ln b, \quad b \geq D.$$

**Proof** This is contained in the second paragraph of the proof of [4, Theorem 2.16]. ■



**Corollary 6.5** *For any  $\delta > 0$ , there exists  $C > 0$  such that*

$$\|(I - \hat{R}^G(ib))^{-1}\|_b \leq C \ln b \text{ for all } b > \delta.$$

**Proof** Let  $A, D$  be as in Lemma 6.4, increased if necessary so that  $A \ln D \geq 1$ .

First suppose that  $b \geq D$ . Let  $m = m(b) = \lceil A \ln b \rceil \geq 1$ . By Lemma 6.4,  $\|\hat{R}^G(ib)^m\|_b \leq \gamma^m \leq \gamma$  and so  $\|(I - \hat{R}^G(ib)^m)^{-1}\|_b \leq (1 - \gamma)^{-1}$ .

Next we use the identity  $(I - \hat{R}^G)^{-1} = (I + \hat{R}^G + \cdots + (\hat{R}^G)^{m-1})(I - (\hat{R}^G)^m)^{-1}$  and Proposition 6.3 to conclude that  $\|(I - \hat{R}^G(ib))^{-1}\|_b \leq mC(1 - \gamma)^{-1} \ll \ln b$ .

To deal with the range  $\delta < b < D$ , we note by the proof of [4, Lemma 2.22] that  $1 \notin \text{spec } \hat{R}^G(ib)$  for any  $b > 0$ . Hence by continuity of the family  $b \mapsto \hat{R}^G(ib)$  and compactness of  $[\delta, D]$  it follows that there is a constant  $C > 0$  such that  $\|(I - \hat{R}^G(ib))^{-1}\|_b \leq C$  for all  $b \in [\delta, D]$ . ■

**Proposition 6.6** *Let  $\epsilon \in (0, \beta)$ . There exists  $C > 0$  such that  $\|\hat{R}^G(i(b+h)) - \hat{R}^G(ib)\|_b \leq Ch^{\beta-\epsilon}$  for all  $b, h > 0$ .*

**Proof** The unnormalized transfer operator has the form

$$P(ib) = \sum_{g \in \mathcal{G}} A_g(b), \quad A_g(b)v = e^{-ib\varphi \circ g} |g'|v \circ g.$$

Now,

$$|A_g(b+h)v - A_g(b)v|_\infty \leq h^{\beta-\epsilon} |\varphi^{\beta-\epsilon} \circ g|_\infty |g'|_\infty |v|_\infty.$$

It follows from the assumptions on  $G$  and  $\varphi$ , together with Proposition 5.7, that  $\sum_{g \in \mathcal{G}} |\varphi^{\beta-\epsilon} \circ g|_\infty |g'|_\infty \ll \int_Z \varphi^{\beta-\epsilon} d\mu_Z < \infty$  and so  $|P(i(b+h)) - P(ib)|_\infty \ll h^{\beta-\epsilon}$ .

Next, the proof of [4, Proposition 2.5] shows that  $|A_g(b)v|_\eta \ll |g'|_\infty \{(1 + |b|^\eta)|v|_\infty + |v|_\eta\}$ . Also,  $A'_g(b) = -i(\varphi \circ g)A_g(b)$ , so

$$\begin{aligned} |A'_g(b)v|_\eta &\leq |\varphi \circ g|_\infty |A_g(b)v|_\eta + |\varphi \circ g|_\eta |A_g(b)v|_\infty \\ &\ll |\varphi \circ g|_\infty |g'|_\infty \{(1 + |b|^\eta)|v|_\infty + |v|_\eta\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |(A_g(b+h)v) - (A_g(b)v)|_\eta &\ll |g'|_\infty \{(1 + |b|^\eta)|v|_\infty + |v|_\eta\} \min\{1, |\varphi \circ g|_\infty h\} \\ &\leq |g'|_\infty \{(1 + |b|^\eta)|v|_\infty + |v|_\eta\} |\varphi \circ g|_\infty^{\beta-\epsilon} h^{\beta-\epsilon}, \end{aligned}$$

and so

$$|P(i(b+h))v - P(ib)v|_\eta \ll \{(1 + |b|^\eta)|v|_\infty + |v|_\eta\} h^{\beta-\epsilon}.$$

It follows from these estimates that  $\|P(i(b+h)) - P(ib)\|_b \leq Ch^{\beta-\epsilon}$ . Finally,  $\hat{R}^G(ib) = f_0^{-1}P(ib)f_0$  where  $f_0, f_0^{-1} \in C^\eta$  and the result follows. ■

**Proposition 6.7**  $\|\hat{B}(ib)1_{\hat{\mathcal{Y}}}\pi_*\|_{C^\eta(\tilde{\mathcal{Y}}) \rightarrow C^\eta(\tilde{\mathcal{Z}})} \leq Cb^\epsilon$  and  $\|\hat{B}(i(b+h))1_{\hat{\mathcal{Y}}}\pi_* - \hat{B}(ib)1_{\hat{\mathcal{Y}}}\pi_*\|_{C^\eta(\tilde{\mathcal{Y}}) \rightarrow C^\eta(\tilde{\mathcal{Z}})} \leq Cb^\epsilon h^{\beta-2\epsilon}$ , for all  $t_0 > 0$ ,  $0 < h < \delta < b$ .

**Proof** This is almost identical to the argument in Section 5. The only difference is that in the final arguments in the proof of Lemma 5.4(c), we have  $g' = p \circ g$  and the estimate  $|p(z) - p(z')| \ll \mu(a)d_Z(Gz, Gz')$  for  $a \in \alpha$ ,  $z, z' \in \alpha$  is replaced by  $|g'z - g'z'| \ll |g'z||z - z'|^\eta$  which holds for all  $z, z' \in Z$ . ■

**Proof of Theorem 6.1** It follows from Corollary 6.5 and Proposition 6.6 (with  $\eta$  replaced by  $\epsilon\eta$ ) that

$$\|(I - \hat{R}^G(i(b+h)))^{-1} - (I - \hat{R}^G(ib))^{-1}\|_b \ll (\ln b)^2 h^{\beta-\epsilon}.$$

Since  $\|v\|_\infty \leq \|v\|_b \leq \|v\|_\eta$ ,

$$\|(I - \hat{R}^G(i(b+h)))^{-1} - (I - \hat{R}^G(ib))^{-1}\|_{C^{\epsilon\eta}(Z) \rightarrow L^\infty(Z)} \ll (\ln b)^2 h^{\beta-\epsilon}.$$

Similarly,  $\|(I - \hat{R}^G(i(b+h)))^{-1}\|_{C^{\epsilon\eta}(Z) \rightarrow L^\infty(Z)} \ll \ln b$ . As in [30], we obtain that

$$\|\hat{T}^G(ib)\|_{C^{\epsilon\eta}(\tilde{Z}) \rightarrow L^\infty(\tilde{Z})} \ll \ln b, \quad \|\hat{T}^G(i(b+h)) - \hat{T}^G(ib)\|_{C^{\epsilon\eta}(\tilde{Z}) \rightarrow L^\infty(\tilde{Z})} \ll (\ln b)^2 h^{\beta-\epsilon}.$$

We now proceed exactly as in Section 5, with Lemma 5.4(c) replaced by Proposition 6.7, to conclude that Lemma 3.3 holds with  $\omega \in (0, 1)$ . (In fact,  $\omega$  can be chosen arbitrarily small.) Hence in the proof of Lemma 3.6, we can take  $m = 2$ . ■

## 7 Verification of hypotheses for Example 1.1

In this section, we return to Example 1.1 and show that Theorem 2.4 applies in this case under the assumption that  $\tau_0$  is both of bounded variation and Hölder continuous.

The first step is to pass from the original suspension semiflow on  $X^{\tau_0}$  to a suspension of the form  $Y^\tau$  where  $Y$  is a probability space and  $\tau$  is a nonintegrable roof function.

We take  $Y$  to be the interval of domain of the rightmost branch of  $f$ . Define  $F = f^{\sigma_0} : Y \rightarrow Y$  for  $\sigma_0 = \min\{n \geq 1 : f^n y \in Y\}$ . Then  $\mu = (\mu_X|Y)/\mu_X(Y)$  is an acip for  $F$ . The corresponding roof function  $\tau : Y \rightarrow \mathbb{R}^+$  is given by  $\tau(y) = \sum_{\ell=0}^{\sigma_0(y)-1} \tau_0(f^\ell y)$ . Let  $F_t : Y^\tau \rightarrow Y^\tau$  be the corresponding suspension semiflow with infinite invariant measure  $\mu^\tau$ .

In [10, Section 9], taking  $Z = Y$ , a reinduced full branch Gibbs-Markov map  $G = F^\sigma : Z \rightarrow Z$  is obtained, with inducing time  $\sigma : Z \rightarrow \mathbb{Z}^+$  having exponential tails. Let  $\mu_Z$  denote the unique acip.

**Proposition 7.1** *Suppose that  $\tau_0$  is Hölder with exponent  $\eta \in (0, 1)$ . Then  $\mu(\tau > t) = ct^{-\beta} + O(n^{-(1+\beta)\eta})$  for some  $c > 0$ .*

**Proof** Let  $\sigma_Z(y) = \sum_{j=0}^{\sigma(y)-1} \sigma_0(F^j y)$ . By [10, Lemma 9.1], the tails of  $\sigma_Z$  satisfy

$$\mu_Z(\sigma_Z > n) = c_Z n^{-\beta} + O(n^{-2\beta}) \text{ for some } c_Z > 0.$$

By [10, Lemma 9.2],  $\mu(\sigma_0 > n) = \mu_Z(\sigma_Z > n) + O(n^{-(1+\beta)})$ , and hence

$$\mu(\sigma_0 > n) = c_F n^{-\beta} + O(n^{-2\beta}) \text{ for some } c_F > 0.$$

This corresponds to [30, Condition (2.3) in the proof of Proposition 2.7] and we can now proceed as in the remainder of the proof of [30, Proposition 2.7].  $\blacksquare$

The map  $F$  is uniformly expanding, but nonMarkov for  $c_1 \in (0, 1)$ . Hence we take  $\mathcal{B}(Y) = BV(Y)$  with norm  $\|v\| = |v|_\infty + \text{Var}(v)$  where  $\text{Var}$  denotes the variation on  $Y$ . In particular,  $\mathcal{B}(Y)$  is embedded in  $L^\infty(Y)$ .

**Proposition 7.2** *Suppose that  $\tau_0$  is of bounded variation and  $C^\eta$  for some  $\eta > 0$ . Then conditions (H1)–(H3) are satisfied.*

**Proof** (H1): It is standard that  $\mathcal{B}(Y) = BV(Y)$  is compactly embedded in  $L^1(Y)$  and that the Lasota-Yorke inequality holds for  $s = 0$ .

Let  $I \subset Y$  be the domain of a branch of  $F$ , and let  $V_I$  denote the variation of  $\tau$  on  $I$ . Since the images  $f^\ell I$  are disjoint for  $\ell = 0, 1, \dots, \sigma_0(I) - 1$ , it follows that  $V_I(\tau) \leq \text{Var}(\tau_0)$ . Hence  $\text{Var}(e^{-s\tau} - 1) \leq \sqrt{2}|s| \text{Var}(\tau_0)$  for all  $s \in \overline{\mathbb{H}}$ . Also  $|e^{-sx} - 1| \leq 2|x|^q$  for all  $x \in [0, 1]$ , so  $\|e^{-s\tau} - 1\| = O(|s|^q)$  for all  $q \in [0, \beta)$ . Hence  $\|\hat{R}(s) - \hat{R}(0)\| \ll |s| + |s|^q$ . Now apply Remark 2.2.

(H2): The properties of reinduced map  $G$  are given in [10, Section 9]. For instance, (H2)(ii) follows from uniform expansion of  $F$ , and (H2)(iii) from uniform expansion of  $F$  combined with Adler's condition.

(H3): Since  $f' \geq 1$  and  $\tau_0 \geq 2$ ,

$$\begin{aligned} |\tau(z) - \tau(z')| &\leq |\tau_0|_\eta \sum_{\ell=0}^{\sigma_0(z)-1} |f^\ell z - f^\ell z'|^\eta \\ &\leq |\tau_0|_\eta \sigma_0(z) |Fz - Fz'|^\eta \leq \frac{1}{2} |\tau_0|_\eta \tau(z) |Fz - Fz'|^\eta, \end{aligned}$$

for all  $z, z' \in a$ ,  $a \in \alpha$ . Hence for  $1 \leq q < \sigma(a)$ ,

$$\begin{aligned} |\tau_q(z) - \tau_q(z')| &\leq \sum_{j=0}^{q-1} |\tau(F^j z) - \tau(F^j z')| \leq |\tau_0|_\eta \sum_{j=0}^{\sigma(a)-1} \tau(F^j z) |F^{j+1} z - F^{j+1} z'|^\eta \\ &\leq |\tau_0|_\eta \sum_{j=0}^{\sigma(a)-1} \tau(F^j z) |Gz - Gz'|^\eta = |\tau_0|_\eta \inf_a \varphi |Gz - Gz'|^\eta. \end{aligned}$$

This is condition (H3).  $\blacksquare$

By Propositions 7.1 and 7.2, we can apply Theorem 2.6 with  $\mathcal{B}(Y) = BV(Y)$  and  $q = (1 + \beta)\eta$  whenever (H4) is satisfied. If we suppose moreover that  $\tau_0$  is  $C^1$  and satisfies (UNI), then the improved result in Theorem 6.1 applies with  $q = 1 + \beta$ .

## A Correlation function of a suspension semiflow

In this appendix, we verify formula (4.1) following [32].

**Proposition A.1**  $\hat{\rho}_{v,w}(s) = \hat{J}(s) + \int_Y \hat{T}(s) v_s w_s d\mu$  for all  $s \in \mathbb{H}$ .

**Proof** First observe that  $\rho_{v,w}(t) = \sum_{n=0}^{\infty} K_n(t)$ , where

$$\begin{aligned} K_n(t) &= \int_{Y^\tau} 1_{\{\tau_n(y) < t+u < \tau_{n+1}(y)\}} v(y, u) w \circ F_t(y, u) d\mu^\tau \\ &= \int_{Y^\tau} 1_{\{\tau_n(y) < t+u < \tau_{n+1}(y)\}} v(y, u) w(F^n y, t+u - \tau_n(y)) d\mu^\tau. \end{aligned}$$

For all  $n \geq 0$ ,

$$\hat{K}_n(s) = \int_0^\infty e^{-st} \int_{Y^\tau} 1_{\{\tau_n(y)-u < t < \tau_{n+1}(y)-u\}} v(y, u) w(F^n y, t+u - \tau_n(y)) d\mu^\tau dt.$$

When  $n \geq 1$ , we have  $u < \tau(y) \leq \tau_n(y)$  for all  $(y, u) \in Y^\tau$ , and hence

$$\hat{K}_n(s) = \int_Y \int_0^{\tau(y)} \int_{\tau_n(y)-u}^{\tau_{n+1}(y)-u} e^{-st} v(y, u) w(F^n y, t+u - \tau_n(y)) dt du d\mu.$$

The substitution  $u' = t + u - \tau_n(y)$  yields

$$\begin{aligned} \hat{K}_n(s) &= \int_Y \left( \int_0^{\tau(y)} e^{su} v(y, u) du \right) \left( \int_0^{\tau(F^n y)} e^{-su'} w(F^n y, u') du' \right) e^{-s\tau_n(y)} d\mu \\ &= \int_Y e^{-s\tau_n} v_s w_s \circ F^n d\mu = \int_Y \hat{R}(s)^n v_s w_s d\mu. \end{aligned}$$

Also,

$$\begin{aligned} \hat{K}_0(s) &= \int_Y \int_0^{\tau(y)} \int_0^{\tau(y)-u} e^{-st} v(y, u) w(y, t+u) dt du d\mu \\ &= \left( \int_Y \int_0^{\tau(y)} \int_0^{\tau(y)} - \int_Y \int_0^{\tau(y)} \int_0^u \right) e^{su} v(y, u) e^{-st} w(y, t) dt du d\mu \\ &= \int_Y v_s w_s d\mu + \hat{J}(s). \end{aligned}$$

Hence  $\hat{\rho}_{v,w}(s) = \hat{J}(s) + \sum_{n=0}^{\infty} \int_Y \hat{R}(s)^n v_s w_s d\mu = \hat{J}(s) + \int_Y (I - \hat{R}(s))^{-1} v_s w_s d\mu$ .  $\blacksquare$

## B Estimates for $1 - \int_Y e^{-s\tau} d\mu$

In this appendix, we write  $s = a + ib \in \mathbb{C}$ . The following result is required in Section 4:

**Proposition B.1** (a) For all  $\beta \in (0, 1]$ ,

$$1 - \int_Y e^{-s\tau} d\mu \sim c_\beta \tilde{\ell}(1/|s|) s^\beta \text{ as } s \rightarrow 0, \text{ where } c_\beta = \begin{cases} i \int_0^\infty e^{-i\sigma} \sigma^{-\beta} d\sigma & \beta < 1 \\ 1 & \beta = 1 \end{cases}.$$

(b) For  $\beta = 1$ ,  $\operatorname{Re} \left( 1 - \int_Y e^{-ib\tau} d\mu \right) \sim \frac{\pi}{2} \ell(1/|b|) |b|$  as  $b \rightarrow 0$ .

**Proof** This is proved in [29, Lemma 2.4] for  $\beta = 1$ , so we focus on the case  $\beta < 1$ . Let  $G(x) = \mu(\tau > x)$ . Then  $1 - \int_Y e^{-s\tau} d\mu = s \int_0^\infty e^{-sx} (1 - G(x)) dx = sI_C(s) - isI_S(s)$ , where

$$I_C(s) = \int_0^\infty e^{-ax} \cos bx (1 - G(x)) dx, \quad I_S(s) = \int_0^\infty e^{-ax} \sin bx (1 - G(x)) dx.$$

By [28, Proposition 6.2], we have for  $a \geq |b|$  that

$$I_C(s) = \tilde{\ell}(1/a)(1+o(1)) + O(|b|a^{-1}\ell(1/a)) = \tilde{\ell}(1/|s|)(1+o(1)) + O(\ell(1/|s|)) \sim \tilde{\ell}(1/|s|).$$

Similarly, for  $a \leq |b|$ , we have  $I_C(s) = \tilde{\ell}(1/|b|)(1+o(1)) + O(a|b|^{-1}\ell(1/|b|)) \sim \tilde{\ell}(1/|s|)$ . Hence  $I_C(s) \sim \tilde{\ell}(1/|s|)$  as  $s \rightarrow 0$ . Similarly, it follows from [28, Proposition 6.2] that  $|I_S(s)| \ll \ell(1/|s|)$ . Part (a) for  $\beta = 1$  follows immediately from these estimates.

Moreover,  $I_S(ib) \sim \frac{\pi}{2} \ell(1/|b|) \operatorname{sgn} b$  as  $b \rightarrow 0$  by the proof of [28, Lemma 6.8]. Since  $\operatorname{Re} \left( 1 - \int_Y e^{-ib\tau} d\mu \right) = bI_S(ib)$ , part (b) follows.  $\blacksquare$

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